Fixed-Point Definability and Polynomial Time

Martin Grohe

Department of Computer Science
Humboldt University Berlin

Outline

1. The Quest for a Logic Capturing PTIME
2. Fixed-Point Logic with Counting
3. Excluded Minors
4. Fixed-Point Logic with Rank
5. Open Problems

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Fagin's Theorem and Gurevich's Question

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Is there a logic that captures PTIME *?*

Corollary (of Fagin's Theorem)

If no logic captures PTIME*, then* PTIME $\neq$ NP.

Motivation from Database Theory

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Chandra and Harel's question is equivalent to Gurevich's question for a logic capturing PTIME*:*

$$\begin{aligned} \textit{relational database} &\approx \textit{finite structure} \\ \textit{query} &\approx \textit{property of finite structures} \end{aligned}$$

Properties of Structures

Structures are finite and relational

Example

Graphs, Boolean circuits, finite words

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Example

Connectedness of graphs, satisfiability of Boolean circuits, the language $\{a^n b^n \mid n \geq 0\}$

Abstract Logics

A logic L consists of a set of sentences and a semantics that associates a property $\mathcal{P}_\phi$ of finite structures with each sentence ϕ.

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Remark

In addition to the minimal requirement of an isomorphism invariant semantics, logics are usually required to satisfy various regularity conditions and effectiveness conditions.

Logics capturing PTIME

Working Definition

A logic L captures PTIME
if for every property $\mathcal{P}$ of finite structures:

$$\mathcal{P} \text{ definable in L} \iff \mathcal{P} \text{ decidable in PTIME}.$$

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The actual definition is more subtle and requires various effectiveness conditions to exclude pathological examples.

Logics capturing PTIME

Working Definition

A logic L captures PTIME on a class $\mathcal{C}$ of structures
if for every property $\mathcal{P}$ of finite structures:

$$\mathcal{P} \text{ definable in L on } \mathcal{C} \iff \mathcal{P} \text{ decidable in PTIME on } \mathcal{C}.$$

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The actual definition is more subtle and requires various effectiveness conditions to exclude pathological examples.

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Main technical difficulty

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Canonization

If there is a polynomial time algorithm that computes a canonical representation for a given structure from a class $\mathcal{C}$, then there is a logic that captures PTIME on $\mathcal{C}$.

The Immerman-Vardi Theorem

Fixed-Point Logic

FP = first-order logic + inductive definitions

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Immerman-Vardi Theorem (1982)

FP *captures* PTIME *on* $\mathcal{O}$.

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FP cannot count

Lemma (Folklore)

For every FP*-sentence* ϕ *there is a* k *such that for all structures* A, B *with* $|A|, |B| \geq k$ *and only empty relations:*

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2. Proof claim for ϕ' by k-pebble game argument.

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The query

"The universe has even cardinality"

is decidable in PTIME,
but not expressible in FP (by Lemma). □

The CFI-Theorem

Fixed-Point Logic with Counting:
FP+C = FP + Counting Operators

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Fixed-Point Logic with Counting:
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Theorem (Cai, Fürer, Immerman 1992)

FP+C *does not capture* PTIME*.*

Capturing PTIME Almost Everywhere

. . .

Theorem (Hella, Kolaitis, Luosto 1996)

FP+C *captures* PTIME *almost everywhere*.

. . . and Even Somewhere Interesting

Theorem

FP+C *captures* PTIME *on the following classes of graphs:*

1. *Trees (Immerman, Lander 1990)*
2. *Classes of graphs of bounded treewidth (G., Mariño 1999)*
3. *Planar graphs (G. 1998)*
4. *Chordal Line Graphs (G. 2009)*

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Minors and Classes Excluding Minors

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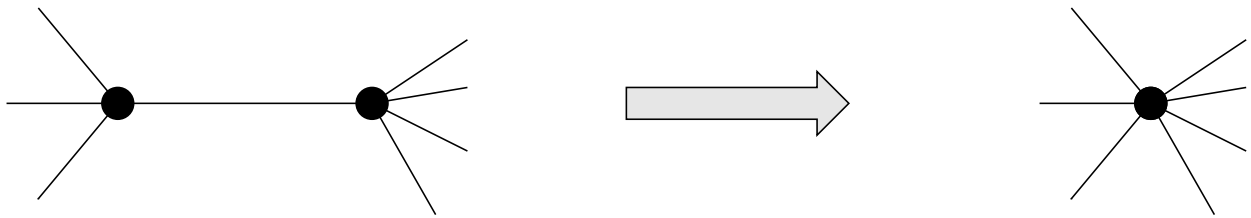
if *G* is obtained from *H* by deleting edges and vertices and contracting edges.



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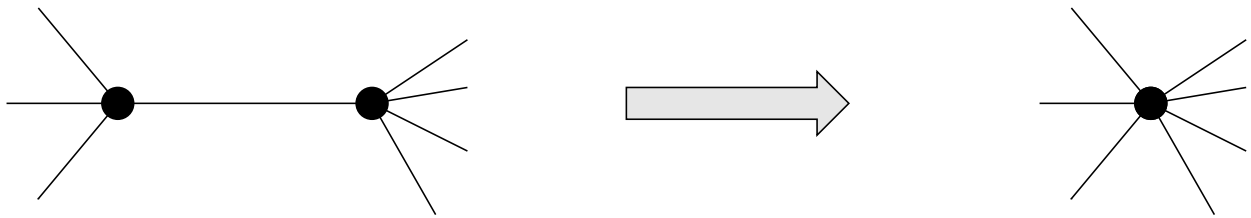

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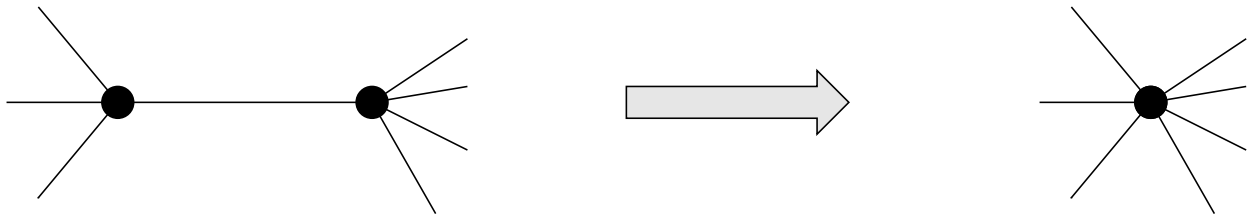

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Example

Classes of bounded treewidth, planar graphs, graphs of bounded genus all exclude minors

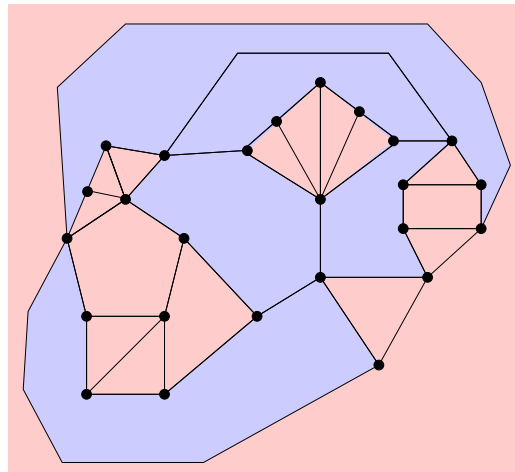
Capturing PTIME on Classes with Excluded Minors

Main Theorem

Let $\mathcal{C}$ be a class of graphs that excludes a minor.
Then FP+C *captures* PTIME *on* $\mathcal{C}$*.*

Proof of the Main Theorem — Step 1:

Definable Orders and 3-Connected Planar Graphs



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Lemma

If structures in class $\mathcal{C}$ admit FP*-definable order (possibly with parameters) then* FP *captures* PTIME *on $\mathcal{C}$.*

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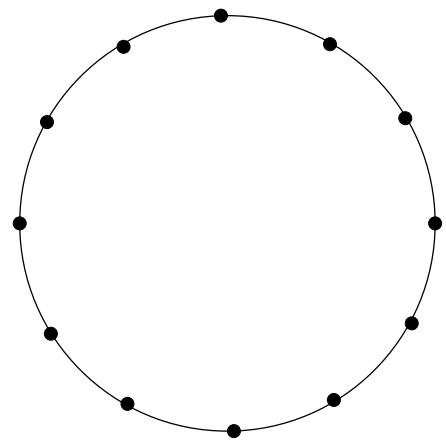
Example

Cycles admit order definable by FP-formula with 2 parameters.



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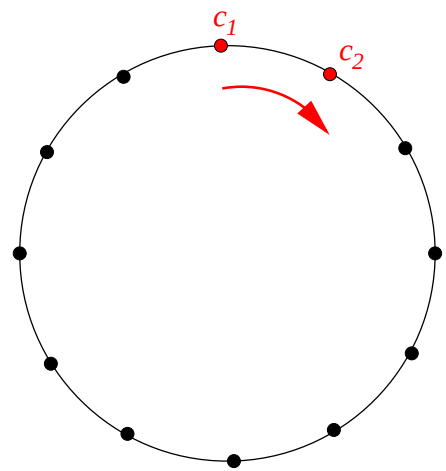

Capturing PTIME on 3-Connected Planar Graphs

Lemma (G. 1998)

3-connected planar graphs admit an FP*-definable order.*

Corollary

FP *captures* PTIME *on the class of 3-connected planar graphs.*

Facial Cycles

Facts

1. *Every planar graph of minimum degree at least 3 has a facial cycle of length at most* 6 *(Folklore, based on Euler's formula)*.

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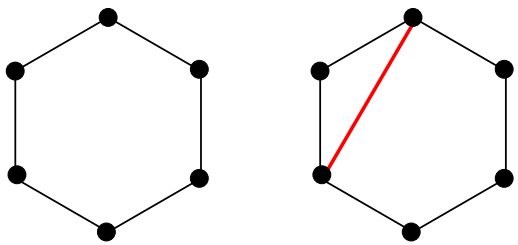
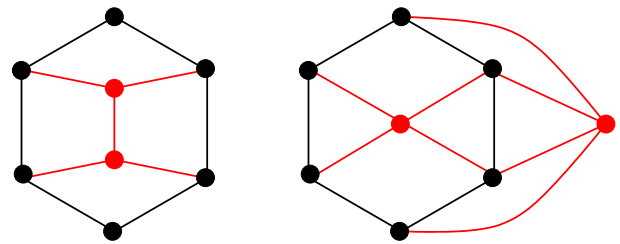
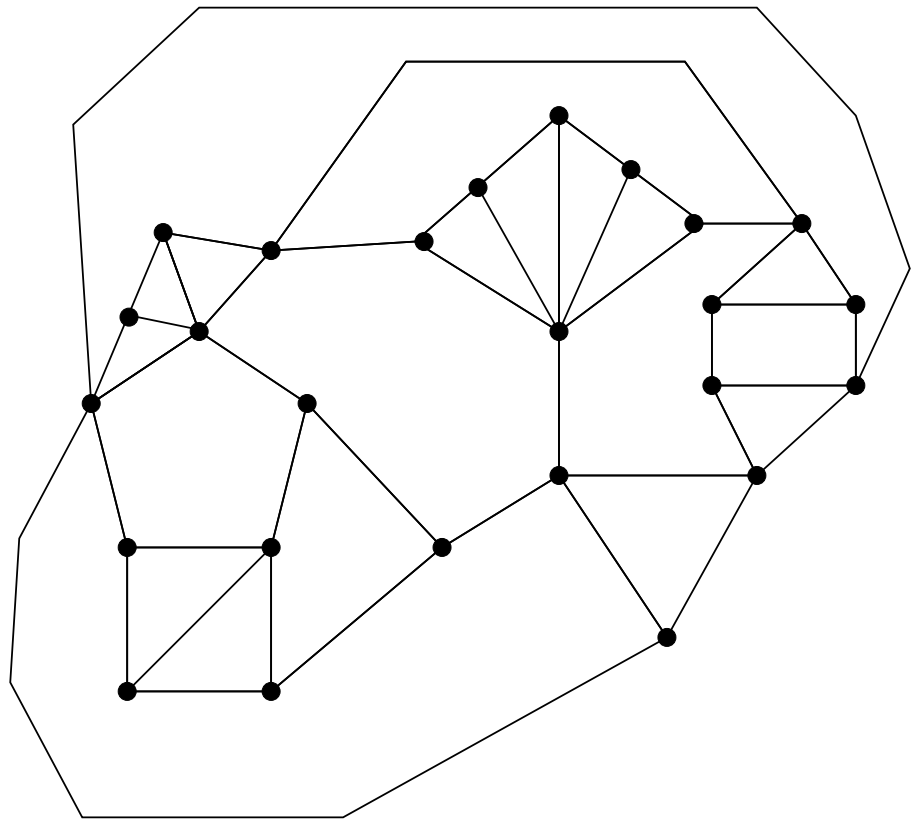


A chordless cycle and a cycle with a chord



A nonseparating and a separating cycle

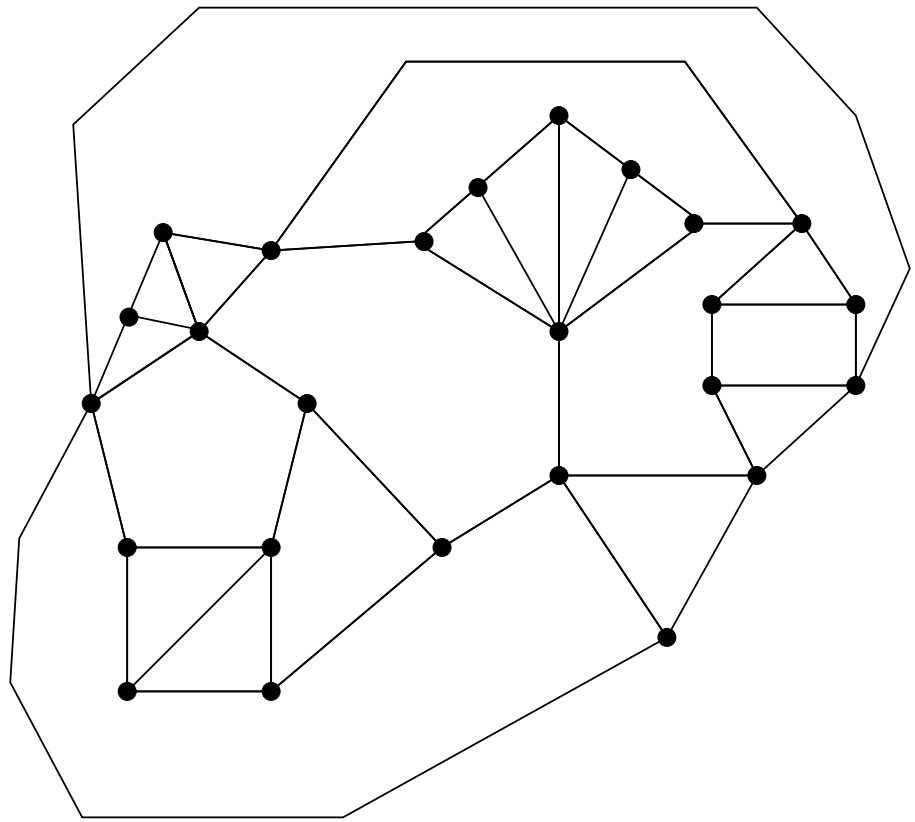
Defining an Order



Defining an Order

Idea

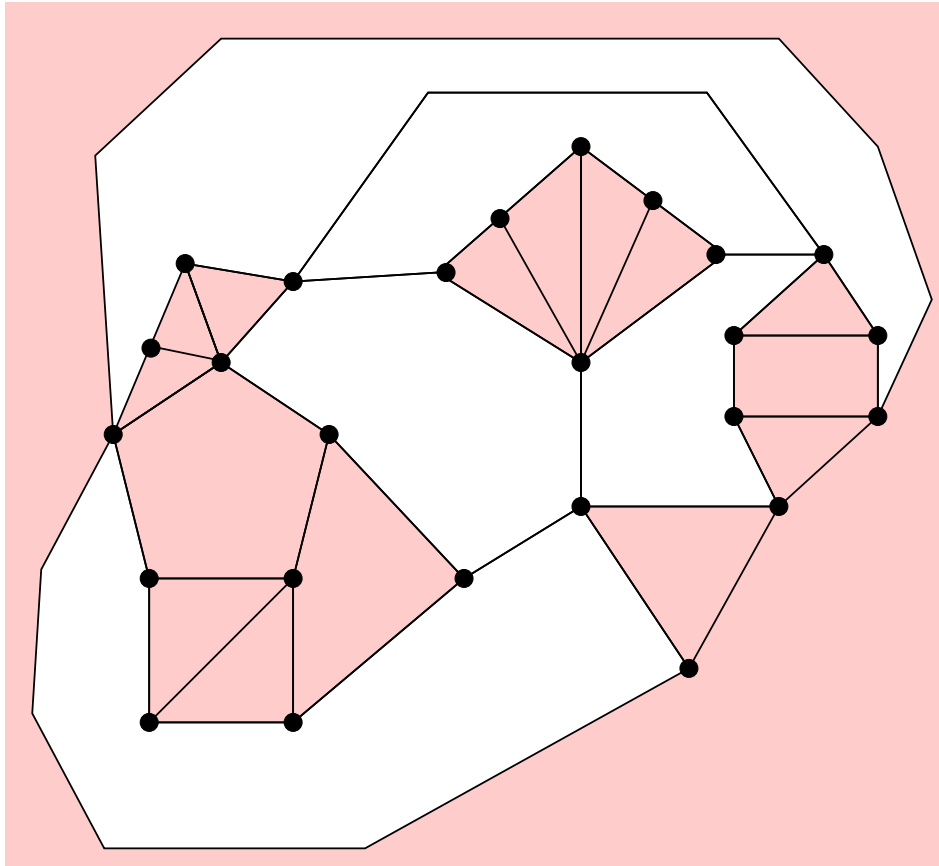
1. Define set of facial cycles



Defining an Order

Idea

1. Define set of facial cycles
   - define facial cycles of length ≤ 6



Defining an Order

Idea

1. Define set of facial cycles
   - define facial cycles of length ≤ 6
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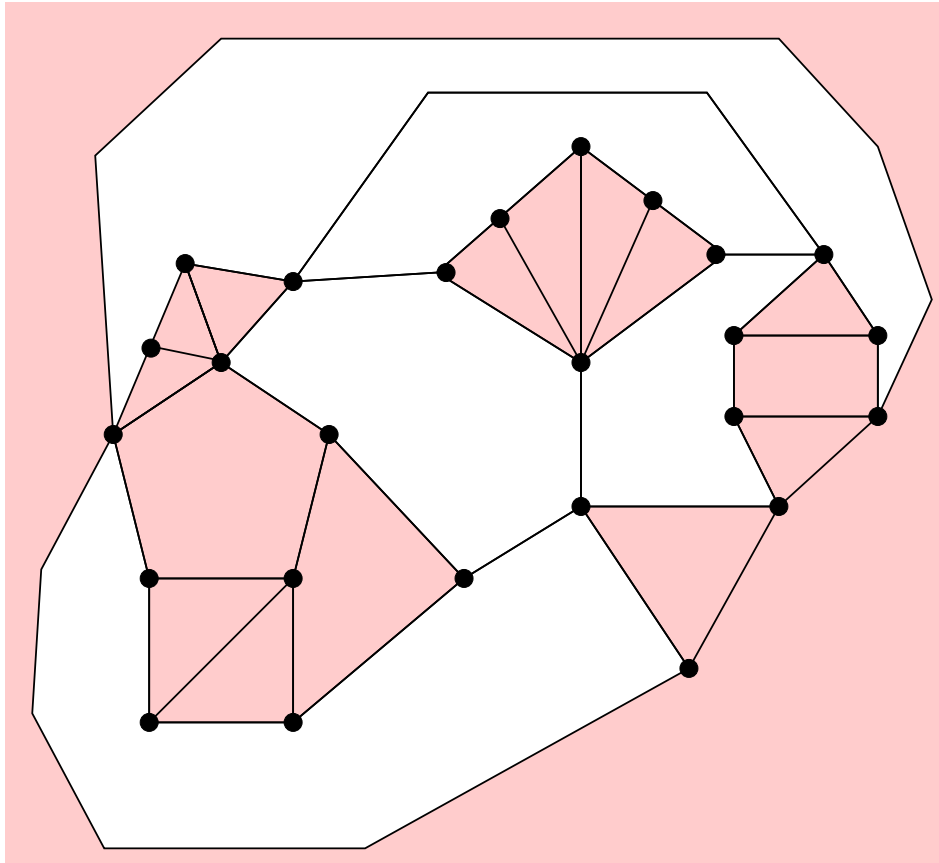

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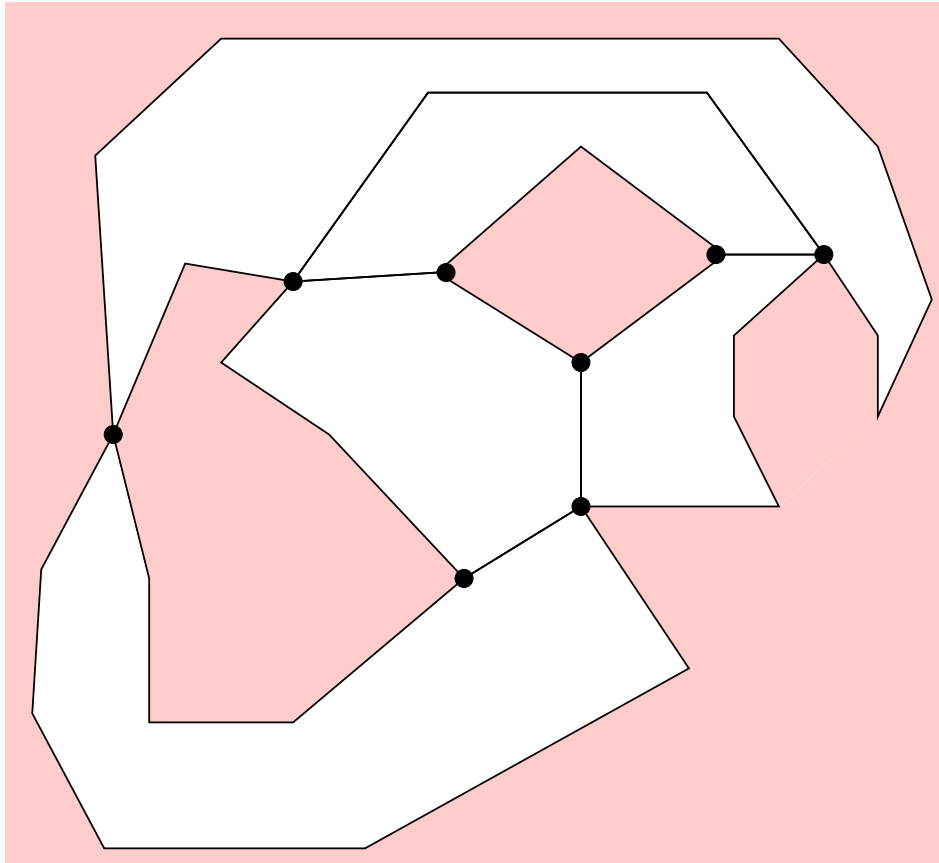

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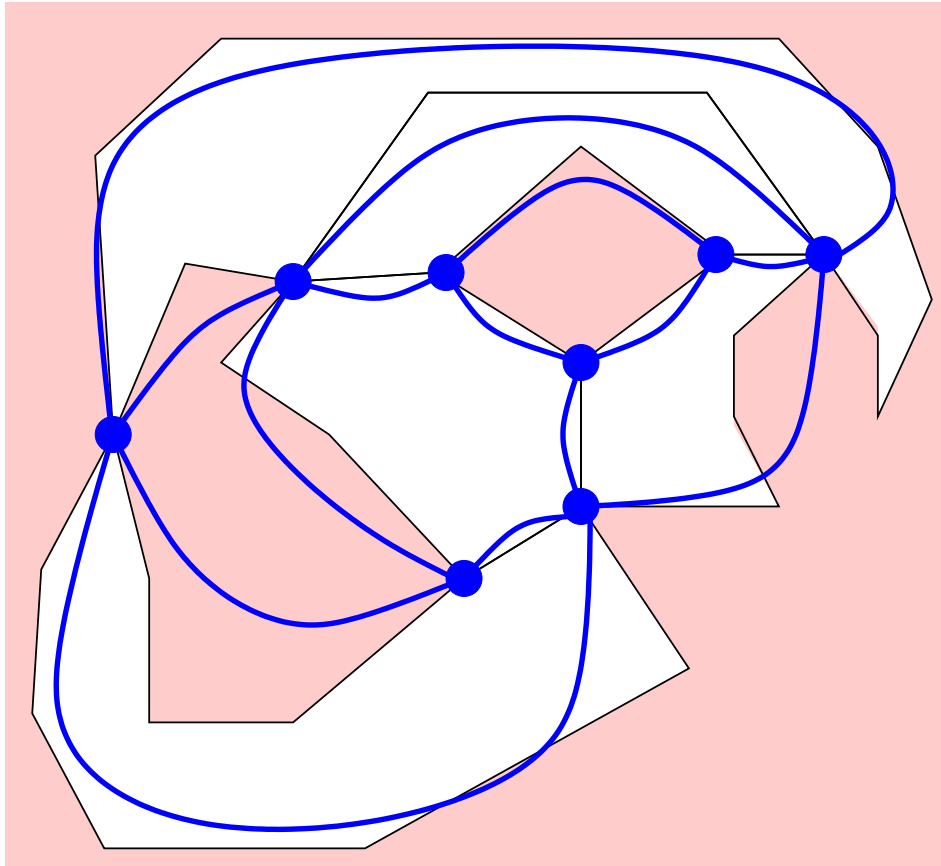

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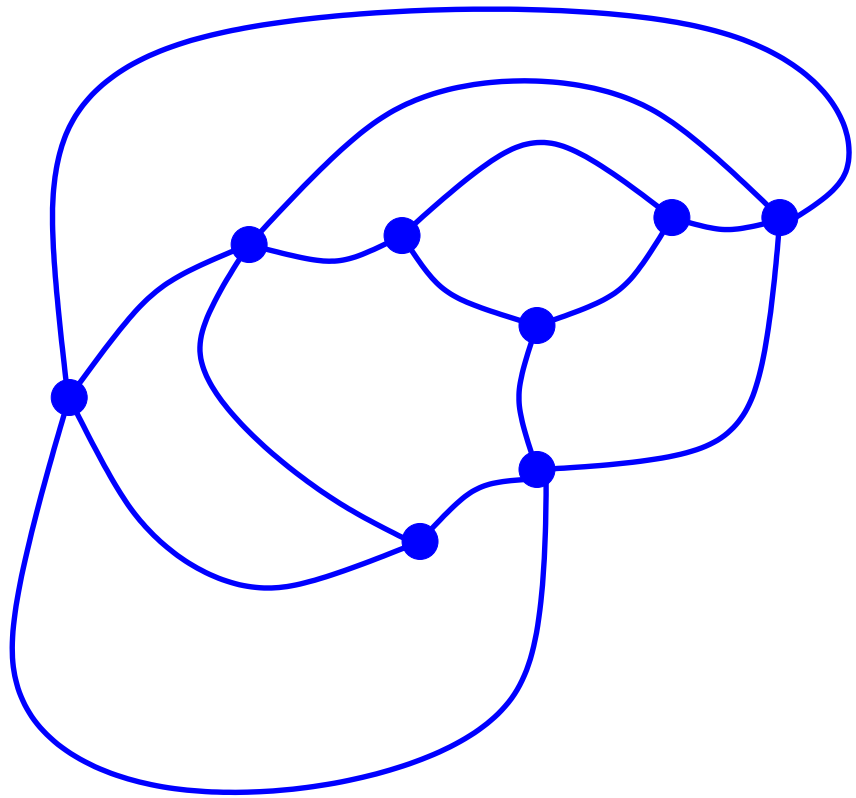

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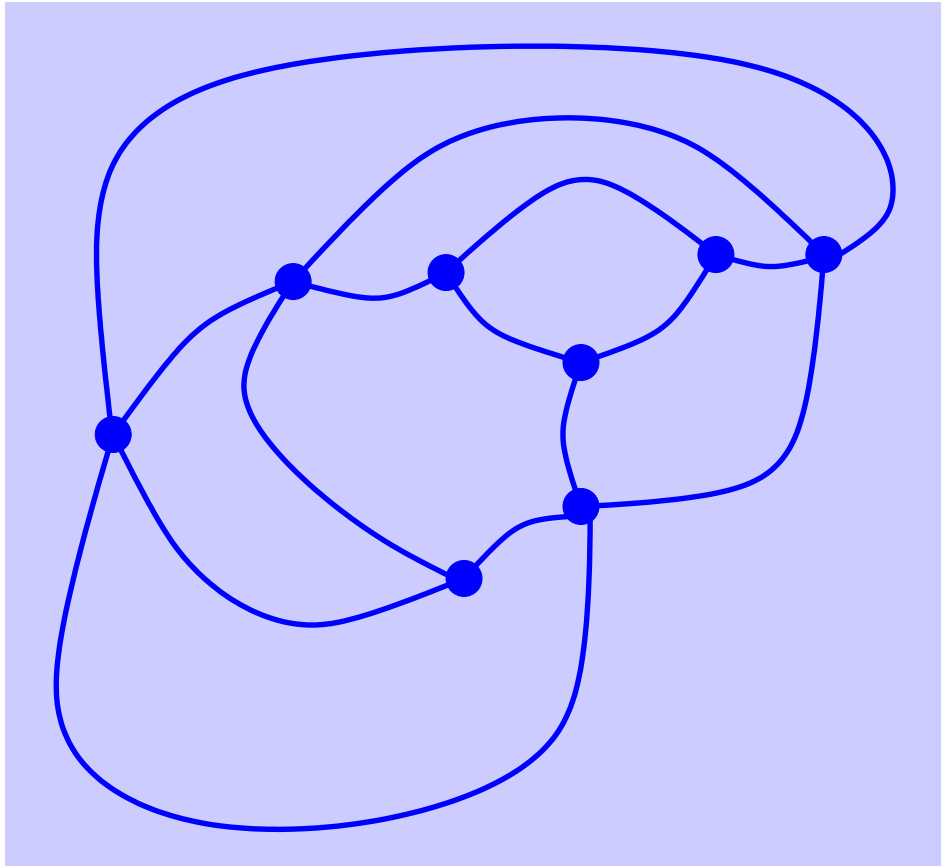

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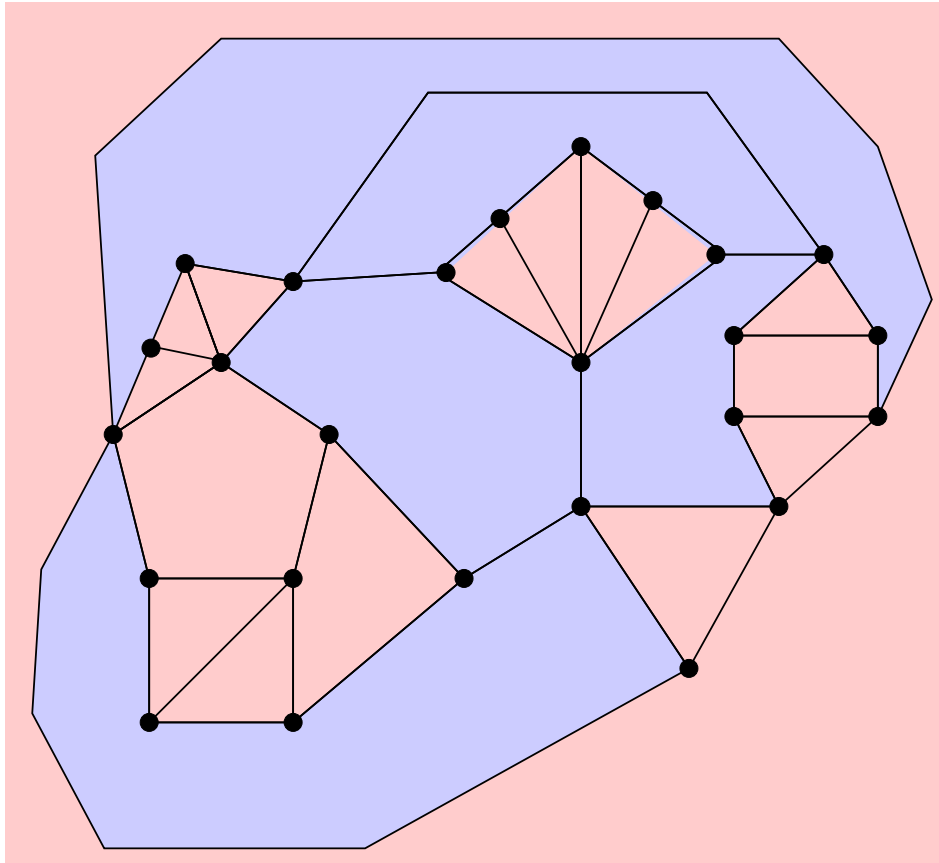

Defining an Order

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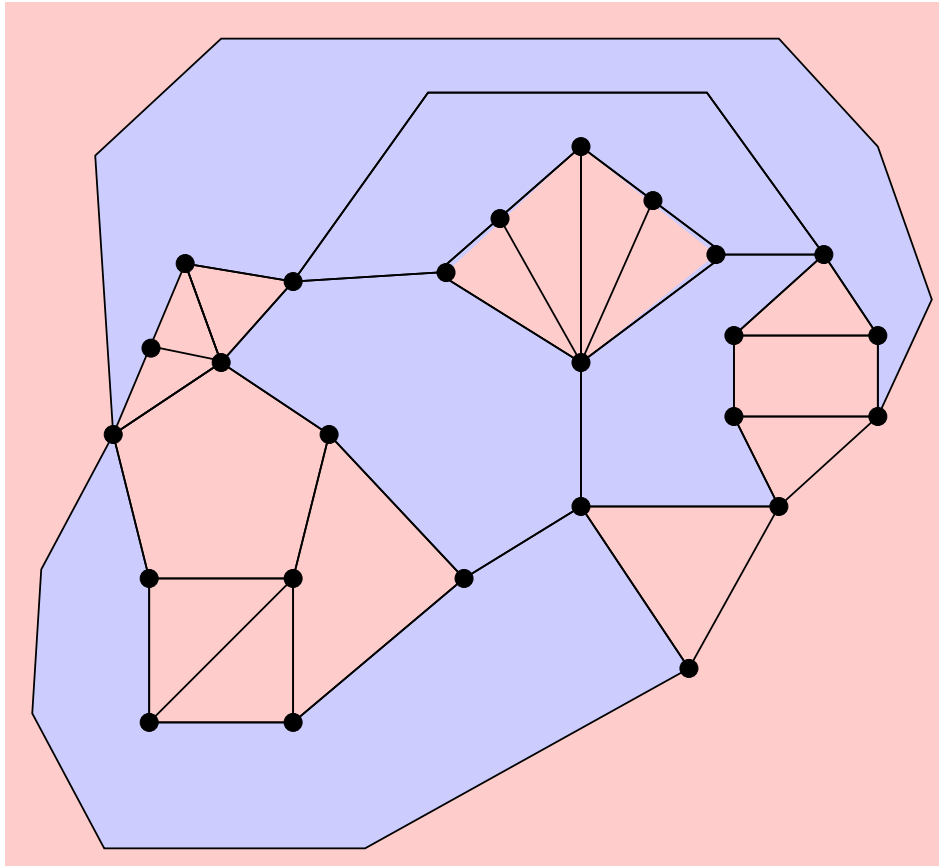
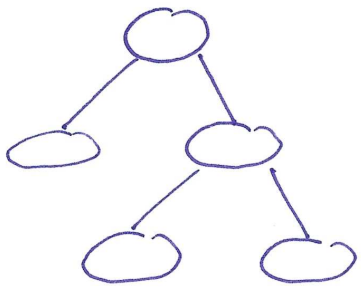
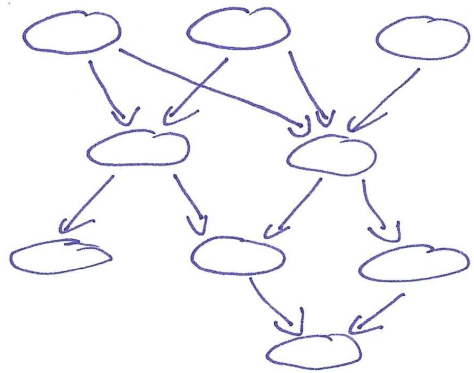
2. use facial cycles to define order



Proof of the Main Theorem — Step 2:

Ordered Treelike Decompositions and Arbitrary Planar Graphs

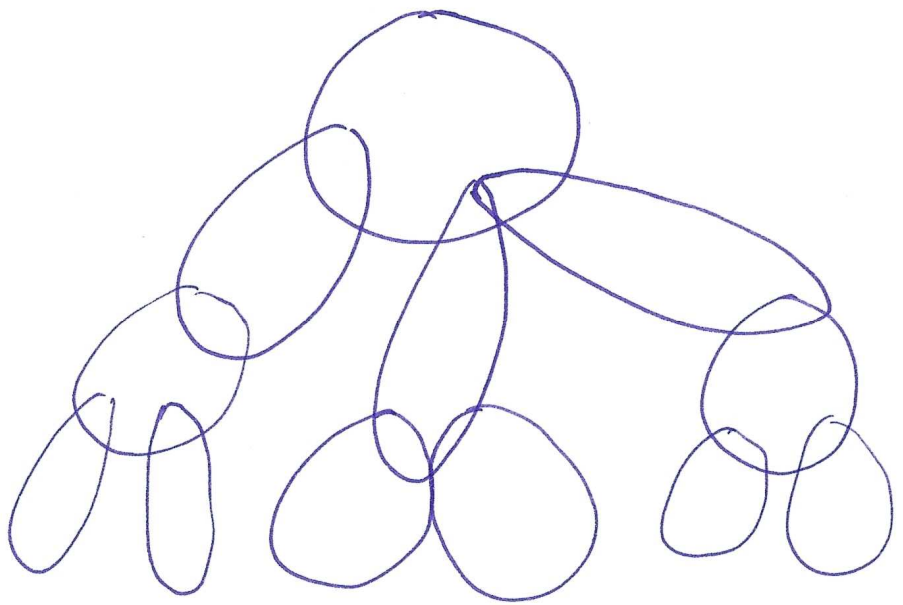




3-Connected Components

Fact

Every graph has a tree decomposition into 3-connected pieces.



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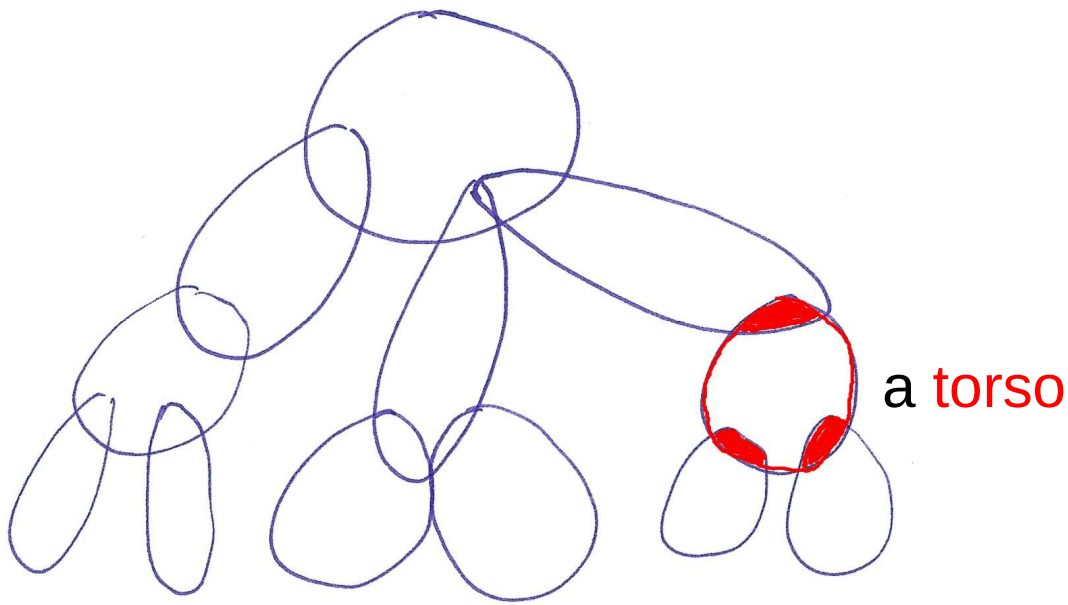
The pieces of the decomposition (formally: torsi) are not necessarily subgraphs, but minors of the original graph.



Treelike Decomposition

Problem

Tree decompositions are not invariant under automorphisms and hence not definable.

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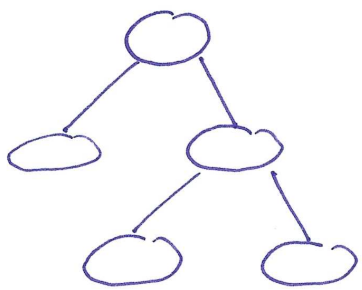
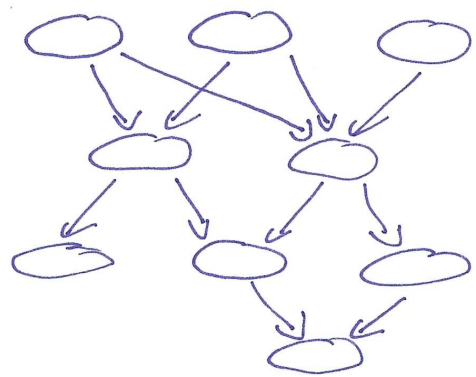
Solution

Use treelike decompositions whose underlying graph is not a tree but a directed acyclic graph.





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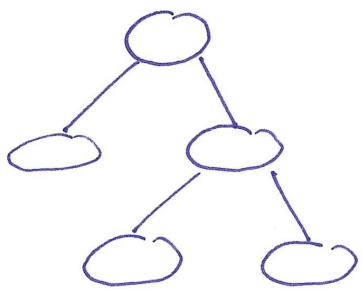

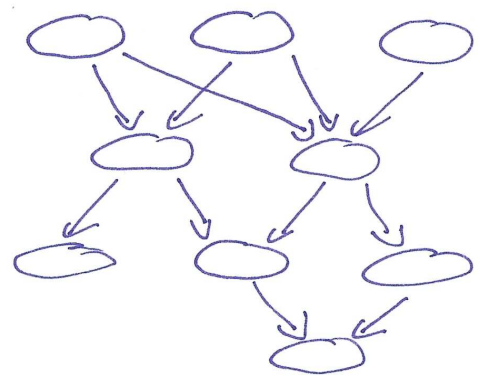

Lemma

Every graph has an FP-definable treelike decomposition into 3-connected torsi that are minors of the graph.

Definable Ordered Treelike Decompositions

Definition

An FP-definable ordered treelike decomposition is an FP-definable treelike decomposition into torsi that admit an FP-definable linear order.

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Corollary

Planar graphs admit FP-definable ordered treelike decompositions.

Capturing PTIME on Planar Graphs

Lemma

Let $\mathcal{C}$ be a class of graphs. If the graphs in $\mathcal{C}$ admit FP+C*-definable ordered treelike decompositions, then* FP+C *captures* PTIME *on $\mathcal{C}$.*

Capturing PTIME on Planar Graphs

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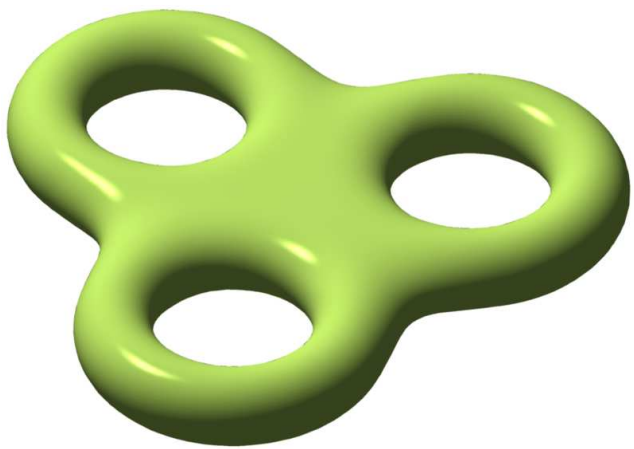
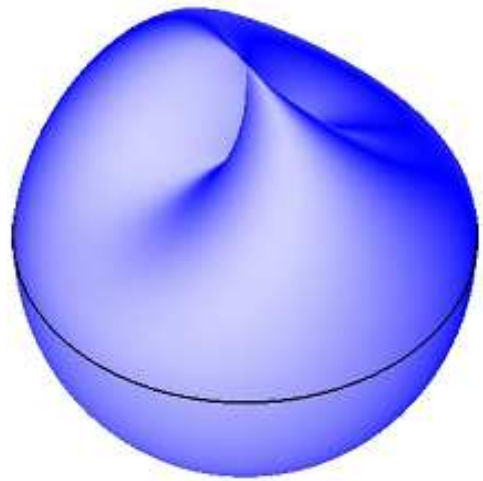
Corollary

FP+C *captures* PTIME *on the class of all planar graphs.*

Proof of the Main Theorem — Step 3:

Graphs of Embeddable in a Surface





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Generalize to arbitrary graphs embeddable into ***S***. □

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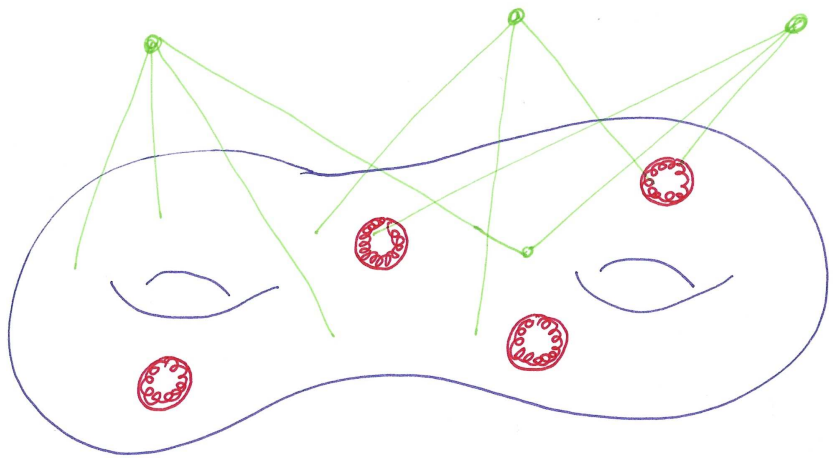
*For every surface **S**, the class of all graphs embeddable into **S** admits FP-definable ordered treelike decompositions.*

Corollary

*For every surface **S**, the logic FP+C capture PTIME on the class of graphs embeddable into **S**.*

Proof of the Main Theorem — Step 4:

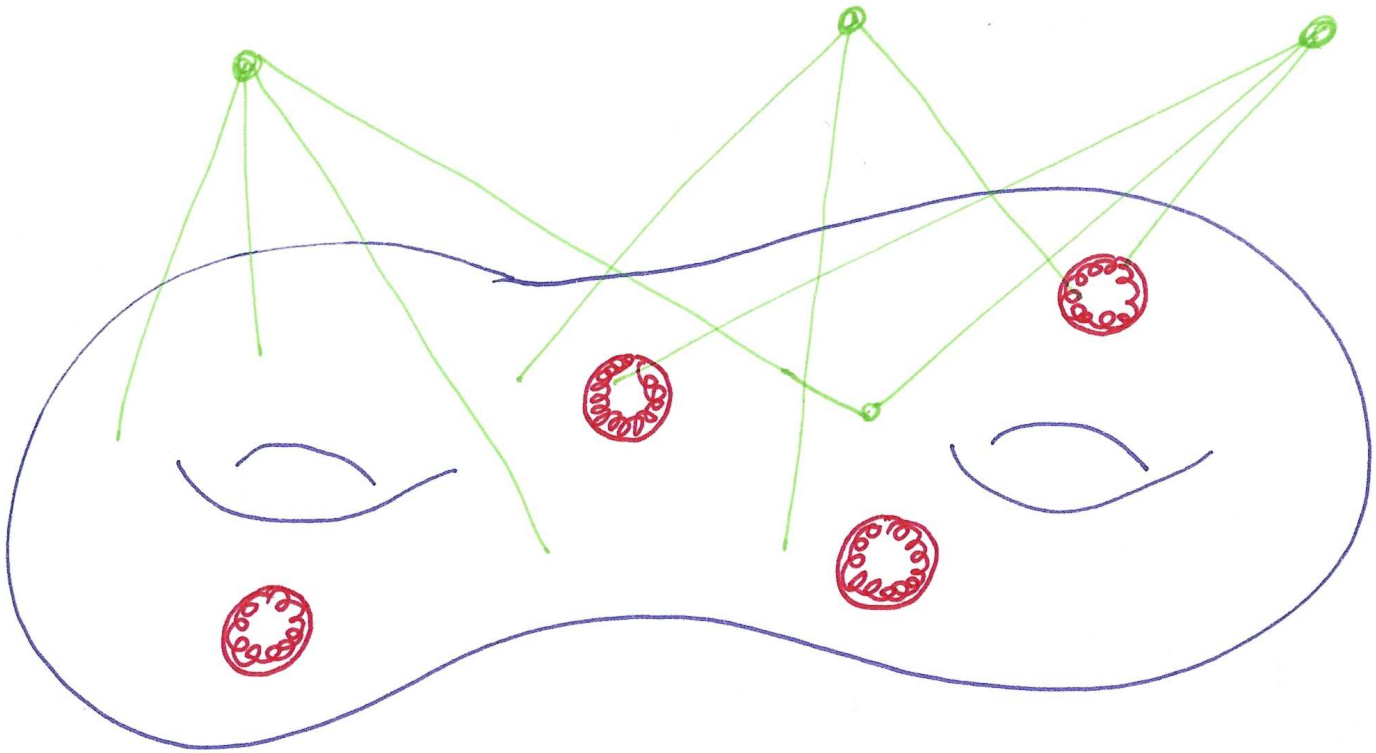
Graphs with Excluded Minors



Almost Embeddable Graph

G almost embeddable into ***S*** if *G* is embeddable into ***S*** up to:

- a bounded number of apices, vertices that have to be removed before the embedding
- a bounded number of vortices, highly nonplanar areas which nevertheless have a regular structure



Robertson and Seymour's Structure Theorem

Theorem (Robertson and Seymour 1984-1999)

*Let $\mathcal{C}$ be a class of graphs with an excluded minor. Then all graphs in $\mathcal{C}$ have tree decompositions into torsi that are almost embeddable into some surface **S**.*

The surface and the bounds on the number of vortices and apices depend on $\mathcal{C}$.

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Every class of graphs with excluded minors admits FP-definable ordered treelike decompositions.

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The Main Theorem follows.

Application to Isomorphism Testing

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Theorem

For every class $\mathcal{C}$ with excluded minors there is a k such that k-WL correctly decides isomorphism of graphs from $\mathcal{C}$.

1. The Quest for a Logic Capturing PTIME
2. Fixed-Point Logic with Counting
3. Excluded Minors
4. Fixed-Point Logic with Rank
5. Open Problems

FP+C cannot do linear algebra

Remember:

CFI-Theorem

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CFI-Theorem
FP+C *does not capture* PTIME.

The proof of the CFI-Theorem exploits the inability of FP+C to do basic linear algebra.
(Atserias, Bulatov, Dawar 2007, Dawar, Richerby, Rossman 2008)

Matrix Rank in FP+C

View formula $\phi(x, y)$ as defining a square $\{0, 1\}$-matrix.

Corollary (of the proof of the CFI-Theorem)

Matrix rank over GF(2) (or any other finite field) is not definable in FP+C*.*

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Theorem (Dawar, G., Holm, Laubner)

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Many interesting open questions.

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Further Classes of Graphs

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Stronger Logics

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Open Problems 6 and 7

Prove that FP+R and CP+C do not capture NP.

Smaller Complexity Classes

Open Problem 8

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Open Problem 9

Is there an effective syntax for order invariant first-order logic ?