Local tree-width, excluded minors, and approximation algorithms

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Abstract

The *local tree-width* of a graph $G = (V, E)$ is the function $\text{ltw}^G : \mathbb{N} \to \mathbb{N}$ that associates with every $r \in \mathbb{N}$ the maximal tree-width of an r-neighborhood in G. Our main graph theoretic result is a decomposition theorem for graphs with excluded minors, which says that such graphs can be decomposed into trees of graphs of almost bounded local tree-width.

As an application of this theorem, we show that a number of combinatorial optimization problems, such as MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, and MAXIMUM INDEPENDENT SET have a polynomial time approximation scheme when restricted to a class of graphs with an excluded minor.

1. Introduction

Tree-width, measuring the similarity of a graph with a tree, has turned out a to be an important notion both in structural graph theory and in the theory of graph algorithms. It is well known that planar graphs may have arbitrarily large tree-width. However, for every fixed d the class of planar graphs of diameter at most d has bounded tree-width. In other words, the tree-width of a planar graph can be bounded by a function of the diameter of the graph. This makes it possible to decompose planar graps into families of graphs of small tree-width in an orderly way. Such decompositions of planar graphs, better known under the name *outerplanar decompositions*, have been explored in various algorithmic settings [5, 8, 13, 11]. The main ideas go back to a fundamental article of Baker [5] on approximation algorithms on planar graphs.

The *local tree-width* of a graph $G = (V, E)$ is the function $\text{ltw}^G : \mathbb{N} \to \mathbb{N}$ that associates with every $r \in \mathbb{N}$ the maximal tree-width of an r-neighborhood in G. More formally, we define the *r-neighborhood* $N_r(v)$ of a vertex $v \in V$ to be the set of all $w \in V$ of distance at most r from v, and we let $\langle N_r(v) \rangle$ denote the subgraph induced by G on $N_r(v)$. Then, denoting the tree-width of a graph H by $\text{tw}(H)$, we let

$$\text{ltw}^G(r) := \max\Big\{\text{tw}\big(\langle N_r(v)\rangle\big) \;\Big|\; v \in V\Big\}.$$

We are mainly interested in classes of graphs of *bounded local tree-width*, that is, classes $\mathcal{C}$ for which there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and

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$r \in \mathbb{N}$ we have $\text{ltw}^G(r) \leq f(r)$. The class of planar graphs is an example. It has been observed by Eppstein [8] that if a class $\mathcal{C}$ is closed under taking minors and has bounded local tree-width (Eppstein calls this the "diameter-treewidth property"), then the graphs in $\mathcal{C}$ admit a decomposition into graphs of small tree-width in the style of the outerplanar decomposition of planar graphs, and the planar-graph algorithms based on this decomposition generalize to graphs in $\mathcal{C}$. Eppstein [9] gave a nice characterization of such classes; he proved that a minor closed class $\mathcal{C}$ of graphs has bounded local tree-width if, and only if, it does not contain all apex graphs.

The main graph-theoretic result of this paper, Theorem 13, can be phrased as follows: Let $\mathcal{C}$ be a minor closed class of graphs that does not contain all graphs. Then all graphs in $\mathcal{C}$ can be decomposed into a tree of graphs that, after removing a bounded number of vertices, have bounded local tree-width. The proof of this result is based on a deep structural characterization of graphs with excluded minors due to Robertson and Seymour [19].

We defer the precise technical statement of our decomposition theorem to Section 4 and turn to its applications now. In this paper, we focus on approximation algorithms, but we shall also use the theorem to re-prove a result of Alon, Seymour, and Thomas [2] that graphs G with an excluded minor have tree-width $O(\sqrt{|G|})$.[1]

Actually, the main result of Alon, Seymour, and Thomas's article is a separator theorem for graphs with an excluded minor, generalizing a well-known separator theorem due to Lipton and Tarjan [14] for planar graphs. These separator theorems have numerous algorithmic applications, among them a polynomial time approximation scheme (PTAS) for the MAXIMUM INDEPENDENT SET problem on planar graphs [15] and, more generally, classes of graphs with an excluded minor [1].

A different approach to approximation algorithms on planar graphs is Baker's [5] technique based on the outerplanar decomposition. It does not only give another PTAS for MAXIMUM INDEPENDENT SET, but also for other problems, such as MINIMUM DOMINATING SET, to which the technique based on the separator theorem does not apply.

We can use our decomposition theorem to extend Baker's approach to arbitrary classes of graphs with an excluded minor. Our purpose here is to explain the technique and not to give an extensive list of problems to which it applies. We show in detail how to get a PTAS for MINIMUM VERTEX COVER on classes of graphs with an excluded minor and then explain how this PTAS has to be modified to solve the problems MINIMUM DOMINATING SET and MAXIMUM INDEPENDENT SET. It should be no problem for the reader to apply the same technique to other optimization problems.

The paper is organized as follows: In Section 2 we fix our terminology and recall a few basic facts about tree-decompositions of graphs. Local tree-width is introduced in Section 3. In Section 4, we prove our decomposition theorem for classes of graphs with an excluded minor. Approximation algorithms are discussed in Section 5, and in Section 6 we briefly explain two other applications of the decomposition theorem.

[1] We have observed this in discussions with Reinhard Diestel and Daniela Kühn.

2. Preliminaries

The vertex set of a graph G is denoted by V^G, the edge set by E^G. Graphs are always assumed to be finite, simple, and undirected. We write $vw \in E^G$ to denote that there is an edge from v to w. For a subset $X \subseteq V^G$, we let $\langle X \rangle^G$ denote the induced subgraph of G with vertex set X. We let $G \setminus X := \langle V^G \setminus X \rangle^G$. For graphs G and H, we let $G \cup H := (V^G \cup V^H, E^G \cup E^H)$. We often omit the superscript ^{G} if G is clear from the context.

K_n denotes the complete graph with n vertices, and for an arbitrary set X, K_X denotes the complete graph with vertex set X. A vertex set $X \subseteq V^G$ in a graph G is a *clique* if $K_X \subseteq G$. The *clique number* $\omega(G)$ of a graph G is the maximal size of a clique in G. For a class $\mathcal{C}$ of graphs, we let $\omega(\mathcal{C})$ be the maximum of the clique numbers of all graphs in $\mathcal{C}$, or ∞, if this maximum does not exist.

Note that if $\mathcal{C}$ is closed under taking subgraphs and is not the class of all graphs, then $\omega(\mathcal{C})$ is finite.

2.1. Graph minors. A *minor* of a graph G is a graph H that can be obtained from a subgraph of G by contracting edges; we write $H \preceq G$ to denote that H is a minor of G.

Note that $H \preceq G$ if, and only if, there is a mapping $h : V^H \to \text{Pow}(V^G)$ such that $\langle h(x) \rangle^G$ is a connected subgraph of G for all $x \in V^H$, $h(x) \cap h(y) = \emptyset$ for $x \neq y \in V^H$, and for every edge $xy \in E^H$ there exists an edge $uv \in E^G$ such that $u \in h(x), v \in h(y)$. We say that the mapping h *witnesses* $H \preceq G$ and write $h : H \preceq G$.

A class $\mathcal{C}$ is *minor closed* if, and only if, for all $G \in \mathcal{C}$ and $H \preceq G$ we have $H \in \mathcal{C}$. We call $\mathcal{C}$ non-trivial if it is not the class of all graphs.

A class $\mathcal{C}$ is H*-free* if $H \not\preceq G$ for all $G \in \mathcal{C}$. We then call H an *excluded minor* for $\mathcal{C}$. Note that a class $\mathcal{C}$ of graphs has an excluded minor if, and only if, there is an $n \geq 1$ such that $\mathcal{C}$ is K_n-free. Furthermore, this is equivalent to saying that $\mathcal{C}$ is contained in some non-trivial minor closed class of graphs.

Robertson and Seymour's [16] *Graph Minor Theorem* states that for every minor closed class $\mathcal{C}$ of graphs there is a finite set $\mathcal{F}$ of graphs such that

$$\mathcal{C} = \{G \mid \forall H \in \mathcal{F} : \ H \not\preceq G\}.$$

For a nice introduction to graph minor theory we refer the reader to the last chapter of [6]; a recent survey is [20].

Tree-decompositions. In this paper, we assume trees to be directed from the root to the leaves. If $tu \in E^T$ we call u a *child* of t and t the *parent* of u. The root of a tree T is always denoted by r^T.[2]

A *tree-decomposition* of a graph G is a pair $(T, (B_t)_{t \in V^T})$, where T is a tree and $(B_t)_{t \in V^T}$ a family of subsets of V^G such that $\bigcup_{t \in V^T} \langle B_t \rangle^G = G$ and for every $v \in V^G$

[2]To have rooted directed trees instead of plain (undirected) trees is not relevant for the graph theory, but convenient for the algorithms.

the set $\{t \mid v \in B_t\}$ is connected. The sets B_t are called the *blocks* of the decomposition. The *width* of $(T, (B_t)_{t \in V^T})$ is the number $\max\{\|B_t\| \mid t \in V^T\} - 1$. The *tree-width* of G, denoted by $\text{tw}(G)$, is the minimal width of a tree-decomposition of G.

The following lemma collects a few simple and well-known facts about tree-decompositions:

Lemma 1. *(1) Let $(T, (B_t)_{t \in V^T})$ be a tree-decomposition of a graph G and $X \subseteq V^G$ a clique. Then there is a $t \in V^T$ such that $X \subseteq B_t$.*

(2) Let G, H be graphs such that $V^G \cap V^H$ is a clique in both G and H. Then $\text{tw}(G \cup H) = \max\{\text{tw}(G), \text{tw}(H)\}$.

(3) Let G be a graph and $X \subseteq V^G$. Then $\text{tw}(G) \leq \text{tw}(G \setminus X) + |X|$.

(4) Let G, H be graphs such that $H \preceq G$. Then $\text{tw}(H) \leq \text{tw}(G)$.

Throughout this paper, for a tree-decomposition $(T, (B_t)_{t \in V^T})$ and $t \in T \setminus \{r^T\}$ with parent s we let $A_t := B_t \cap B_s$. We let $A_{r^T} := \emptyset$.

The *adhesion* of $(T, (B_t)_{t \in V^T})$ is the number

$$\text{ad}(T, (B_t)_{t \in V^T}) := \max\{\|A_t\| \mid t \in V^T\}.$$

The *torso* of $(T, (B_t)_{t \in V^T})$ at $t \in V^T$ is the subgraph

$$[B_t] := \langle B_t \rangle^G \cup K_{A_t} \cup \bigcup_{u \text{ child of } t} K_{A_u},$$

or equivalently, the subgraph with vertex set B_t in which two vertices are adjacent if, and only if, either they are adjacent in G or they both belong to a block B_u with $u \neq t$. $(T, (B_t)_{t \in V^T})$ is a tree-decomposition of G *over* a class $\mathcal{B}$ of graphs if all its torsos belong to $\mathcal{B}$.

Note that the adhesion of a tree-decomposition over $\mathcal{B}$ is bounded by $\omega(\mathcal{B})$. Moreover, it can be easily seen that if a graph has a tree-decomposition over a minor-closed class $\mathcal{B}$ then it has a tree-decomposition over $\mathcal{B}$ of adhesion at most $\omega(\mathcal{B}) - 1$.

Path decompositions. A *path-decomposition* of a graph G is a tree decomposition where the underlying tree is a path. Of course we can always assume that the path P of a path decomposition $(P, (B_p)_{p \in P})$ has vertex set $V^P = \{1, \ldots, m\}$, for some $m \in \mathbb{N}$, and that the vertices occur on P in their natural order (that is, we have $i(i+1) \in E^P$ for $1 \leq i < m$).

Lemma 2. *Let G, H be graphs and $(\{1, \ldots, m\}, (B_i)_{1 \leq i \leq m})$ a path-decomposition of H of width k. Let $x_1 \ldots x_m$ be a path in G such that $x_i \in B_i$ for $1 \leq i \leq m$ and $V^G \cap V^H = \{x_1, \ldots, x_m\}$. Then* $\text{tw}(G \cup H) \leq (\text{tw}(G) + 1)(k + 1) - 1$.

Proof: Let $(T, (C_t)_{t \in V^T})$ be a tree-decomposition of G. Then $(T, (C'_t)_{t \in V^T})$ with

$$C'_t = C_t \cup \bigcup_{\substack{1 \leq i \leq m, \\ x_i \in C_t}} B_i$$

is a tree-decomposition of $G \cup H$. The size of the blocks C'_t can easily be bounded by $(\text{tw}(G) + 1)(k + 1)$. □

3. Local tree-width

The distance $d^G(x, y)$ between two vertices x, y of a graph G is the length of the shortest path in G from x to y. For $r \geq 1$ and $x \in G$ we define the *r-neighborhood* around x to be $N_r^G(x) := \{y \in V^G \mid d^G(x, y) \leq r\}$.

Definition 3. (1) The *local tree-width* of a graph G is the function $\text{ltw}^G : \mathbb{N} \to \mathbb{N}$ defined by

$$\text{ltw}^G(r) := \max\bigl\{\text{tw}(\langle N_r^G(x)\rangle) \bigm| x \in V^G\bigr\}.$$

(2) A class $\mathcal{C}$ of graphs has *bounded local tree-width* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\text{ltw}^G(r) \leq f(r)$ for all $G \in \mathcal{C}$, $r \in \mathbb{N}$.

$\mathcal{C}$ has *linear local tree-width* if there is a $\lambda \in \mathbb{R}$ such that $\text{ltw}^G(r) \leq \lambda r$ for all $G \in \mathcal{C}$, $r \in \mathbb{N}$.

Note that the local tree-width of a graph is not minor-monotone (that is, $H \preceq G$ does not imply $\text{ltw}^H(r) \leq \text{ltw}^G(r)$ for all r).

Example 4. Let G be a graph of tree-width at most k. Then $\text{ltw}^G(r) \leq k$ for all $r \in \mathbb{N}$.

Example 5. Let G be a graph of valence at most l, for an $l \geq 1$. Then $\text{ltw}^G(r) \leq l(l-1)^{r-1}$ for all $r \in \mathbb{N}$.

The planar graph algorithms due to Baker and others that we mentioned in the introduction are based on the following result:

Proposition 6 (Robertson and Seymour [17]). *The class of planar graphs has linear local tree-width. More precisely, for every planar graph G and $r \geq 1$ we have* $\text{ltw}^G(r) \leq 3r$.

In this paper, a *surface* is a compact connected 2-manifold with (possibly empty) boundary. The (orientable or non-orientable) *genus* of a surface S is denoted by $g(S)$. An *embedding* of a graph G in a surface S is a mapping Π that associates distinct points of S with the vertices of G and internally disjoint simple curves in S with the edges of G in such a way that for all vertices v and edges e of G, $\Pi(v)$ is not an interior point of the curve $\Pi(e)$, and $\Pi(v)$ is an endpoint of $\Pi(e)$ if, and only if, v is incident with e.

Proposition 7 (Eppstein [9]). *Let S be a surface. Then the class of all graphs embeddable in S has linear local tree-width. More precisely, there is a constant c such that for all graphs G embeddable in S and for all $r \geq 0$ we have* $\text{ltw}^G(r) \leq c \cdot g(S) \cdot r$.

In the next subsection, we prove an extension of Proposition 7 that forms the bases of our decomposition theorem for graphs with excluded minors.

But before we do so, let us state another result due to Eppstein that characterizes the minor closed classes of graphs of bounded local tree-width. An *apex graph* is a graph G that has a vertex $v \in V^G$ such that $G \setminus \{v\}$ is planar.

Theorem 8 (Eppstein [8, 9]). *Let $\mathcal{C}$ be a minor-closed class of graphs. Then $\mathcal{C}$ has bounded local tree-width if, and only if, $\mathcal{C}$ does not contain all apex graphs.*

It is an interesting open problem whether there is a minor closed class of graphs of bounded local tree-width that does not have linear (or polynomially bounded) local tree-width.

Almost embeddable graphs. Let S be a surface with non-empty boundary. The boundary of S consists of finitely many connected components $C_1, \ldots, C_\kappa$, each of which is homeomorphic to the cycle S^1.

We now define a graph G to be *almost embeddable* in S. Roughly, this means that we can obtain G from a graph G_0 embedded in S by attaching at most κ graphs of path-width at most κ to G_0 along the boundary cycles $C_1, \ldots, C_\kappa$ in an orderly way.

This notion plays an important role in the structure theory of graphs with excluded minors, to be outlined in the next subsection.

Definition 9. Let S be a surface with boundary cycles $C_1, \ldots, C_\kappa$. A graph G is *almost embeddable* in S if there are (possibly empty) subgraphs $G_0, \ldots, G_\kappa$ of G such that

– $G = G_0 \cup \ldots \cup G_\kappa$,

– G_0 has an embedding Π in S,

– $G_1, \ldots, G_\kappa$ are pairwise disjoint,

– for $1 \leq i \leq \kappa$, G_i has a path decomposition $(\{1, \ldots, m_i\}, (B^i_j)_{1 \leq j \leq m_i})$ of width at most κ,

– for $1 \leq i \leq \kappa$ there are vertices $x^i_1, \ldots, x^i_{m_i} \in V^{G_0}$ such that $x^i_j \in B^i_j$ for $1 \leq j \leq m_i$ and $V^{G_0} \cap V^{G_i} = \{x^i_1, \ldots, x^i_{m_i}\}$,

– for $1 \leq i \leq \kappa$, we have $\Pi(V^{G_0}) \cap C_i = \{\Pi(x^i_1), \ldots, \Pi(x^i_{m_i})\}$, and the points $\Pi(x^i_1), \ldots, \Pi(x^i_{m_i})$ appear on C_i in this order (either if we walk clockwise or anti-clockwise).

Proposition 10. *Let S be a surface. Then the class of all graphs almost embeddable in S has linear local tree-width.*

Proof: Let G be a graph that is almost embeddable in S. We use the notation of Definition 9. Let H_0 be the graph obtained from G_0 by adding new vertices $z_1, \ldots, z_\kappa$, and edges (z_i, x^i_j), (x^i_j, x^i_{j+1}), and (x^i_κ, x^i_1), for $1 \leq i \leq \kappa$, $1 \leq j \leq m_i$ (see Figure 1). Clearly, H_0 is still embeddable in S. For $1 \leq i \leq \kappa$ we let $H_i := H_0 \cup G_1 \cup \ldots \cup G_i$.

Let $\lambda \in \mathbb{N}$ such that for every graph G embedabble in S and every $r \in \mathbb{N}$ we have $\text{ltw}^G(r) \leq \lambda r$ (such a λ exists by Proposition 7). For $r \in \mathbb{N}$ we let $f_0(r) := \lambda r$ and, for $i \in \mathbb{N}$, we let $f_i(r) := (f_{i-1}(r+1)+1)(\kappa+1)-1$. Then f_i is a linear function for every $i \in \mathbb{N}$.

By induction on $i \geq 0$ we shall prove that for every $r \in \mathbb{N}$ and $x \in V^{H_i}$ we have

$$\text{tw}\big(\langle N^{H_i}_r(x)\rangle\big) \leq f_i(r). \tag{1}$$

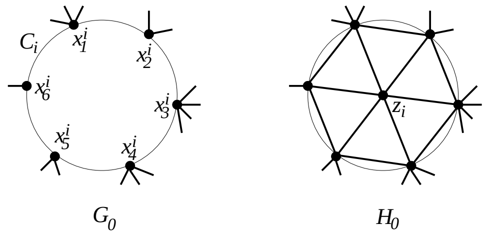

Figure 1. From G_0 to H_0

For $i = 0$, this is immediate. So we assume that $i \geq 1$ and that we have proved (1) for $i - 1$.

For all $x \in H_i$, we either have $N_r^{H_i}(x) \subseteq H_{i-1}$, or $N_r^{H_i}(x) \subseteq G_i$, or $N_r^{H_i} \cap \{x_1^i, \dots, x_{m_i}^i\} \neq \emptyset$.

If $N_r^{H_i}(x) \subseteq V^{H_{i-1}}$ then $\text{tw}\big(\langle N_r^{H_i}(x)\rangle^{H_i}\big) \leq f_{i-1}(r) \leq f_i(r)$.

If $x \in V^{H_{i-1}}$ and $N_r^{H_i}(x) \not\subseteq V^{H_{i-1}}$, then $N_{r-1}^{H_i}(x) \cap \{x_1^i, \dots, x_{m_i}^i\} \neq \emptyset$. By the construction of H_0, this implies $z_i \in N_r^{H_{i-1}}(x)$ and thus $\{x_1^i, \dots, x_{m_i}^i\} \subseteq N_{r+1}^{H_{i-1}}(x)$.

By Lemma 2 and the induction hypothesis we get

$$\begin{aligned} \text{tw}\big(\langle N_r^{H_i}(x)\rangle^{H_i}\big) &\leq \text{tw}\big(\langle N_{r+1}^{H_{i-1}}(x) \cup V^{G_i}\rangle^{H_i}\big) \\ &\leq (f_{i-1}(r+1)+1)(\kappa+1) - 1 = f_i(r). \end{aligned}$$

If $x \in V^{G_i}$, then $N_r^{H_i}(x) \cap V^{H_{i-1}} \subseteq N_{r+1}^{H_{i-1}}(z_i)$. Thus by Lemma 2 and the induction hypothesis we have

$$\begin{aligned} \text{tw}\big(\langle N_r^{H_i}(x)\rangle^{H_i}\big) &\leq \text{tw}\big(\langle N_{r+1}^{H_{i-1}}(z_i) \cup V^{G_i}\rangle^{H_i}\big) \\ &\leq (f_{i-1}(r+1)+1)(\kappa+1) - 1 = f_i(r). \end{aligned}$$

□

Recall that local tree-width is not minor-monotone. However, we do have

$$H \subseteq G \implies \text{ltw}^H \leq \text{ltw}^G. \tag{2}$$

Proposition 11. *Let S be a surface. Then the class of all minors of graphs almost embeddable in S has linear local tree-width.*

Proof: Recall the proof of Proposition 10. We use the same notation here. Suppose G' is a minor of G. We can assume that G' is a subgraph of a graph G'' obtained from G only by contracting edges. Because of (2) we can even assume that $G' = G''$.

Let $X = \{x_j^i \mid 1 \leq i \leq \kappa, 1 \leq j \leq m_i\}$. Contracting edges with at least one endpoint not in X is unproblematic, because the resulting graph is still almost embeddable in S.

So we can further assume that G' is obtained from G by contracting edges $e_1, \dots,$ e_n with both endpoints in X. Let $H := H_\kappa$ (the graph obtained from G by adding the vertices z_i and corresponding edges as in Figure 1). Let H' be the graph obtained from H by contracting $e_1, \dots, e_n$, and let $h : H' \preceq H$ witness these edge contractions.

The key observation is that for all $x, y \in V^{H'}$ and $u \in h(x), v \in h(y)$ we have

$$d^H(u, v) \le d^{H'}(x, y) + 3\kappa - 1 \tag{3}$$

(no matter how large n is). To see this, let P' be a shortest path from x to y in H'. Let P be a path from u to v in H such that P' is obtained from P by contracting the edges $e_1 \dots, e_n$. Let us call such an edge an (i,j)-edge if it connects a vertex in $\{x^i_1, \dots, x^i_{m_i}\}$ with a vertex in $\{x^j_1, \dots, x^j_{m_j}\}$. Suppose that $P = w_1 \dots w_r$. For $1 \le i \le \kappa$, let w_s and w_t, where $1 \le s \le t \le r$, be the first and last vertex from $\{x^i_1, \dots, x^i_{m_i}\}$ on P. If $s < t$ we replace the interval $w_s \dots w_t$ in P by $w_s z_i w_t$. Doing this for $1 \le i \le \kappa$ we obtain a new path Q from u to v in H. This path Q contains at most 2κ edges that are not on P and no (i,i)-edges. Furthermore, for $1 \le i < j \le n$ the number of (i,j)-edges on Q is at most $(\kappa - 1)$. To see this, assume that Q contains at least κ such edges. Then there would be a "cycle" $i = i_1, i_2, \dots, i_l = i$ such that for $1 \le j < l$, Q contains an (i_j, i_{j+1})-edge. However, this cycle would have been removed while transforming P to Q.

Hence length$(Q) \le$ length$(P') + 3\kappa - 1$, which proves (3).

(3) implies that for all $r \ge 0$, $x \in V^{H'}$, and $u \in h(x)$ we have

$$\langle N^{H'}_r(x)\rangle \preceq \langle N^H_{r+3\kappa-1}(u)\rangle. \tag{4}$$

To see this, let $y \in N^{H'}_r(x)$. Then for all $v \in h(y)$, by (3) we have $v \in N^H_{r+3\kappa-1}(u)$. Thus $h(N^{H'}_r(x)) \subseteq$ Pow$(N^H_{r+3\kappa-1}(u))$. Therefore the restriction of h to $N^{H'}_r(x)$ witnesses $\langle N^{H'}_r(x)\rangle \preceq \langle N^H_{r+3\kappa-1}(u)\rangle$. This proves (4).

By (1) and (4) we get tw$(\langle N^{H'}_r(x)\rangle) \le f_\kappa(r+3\kappa-1)$. The statement of the lemma follows. $\square$

4. Graphs with excluded minors

The following deep structure theorem for K_n-free graphs plays a central role in the proof of the Graph Minor Theorem. For a surface S and $\mu \in \mathbb{N}$ we let $\mathcal{A}(S, \mu)$ be the class of all graphs G such that there is an $X \subseteq V^G$ with $\|X\| \le \mu$ such that $G \setminus X$ is almost embeddable in S.

Theorem 12 (Robertson and Seymour [19]). *For every $n \in \mathbb{N}$ there exist $\mu \in \mathbb{N}$ and surfaces S, S' such that all K_n-free graphs have a tree-decomposition over $\mathcal{A}(S, \mu) \cup \mathcal{A}(S', \mu)$.*

Further details concerning this theorem can be found in [7, 20, 19].

For $\lambda, \mu \geq 0$ we let

$$\mathcal{L}(\lambda) := \left\{ G \mid \forall H \preceq G \; \forall r \geq 0 : \; \mathrm{ltw}^H(r) \leq \lambda \cdot r \right\},$$
$$\mathcal{L}(\lambda, \mu) := \left\{ G \;\middle|\; \exists X \subseteq V^G : \left(\|X\| \leq \mu \wedge G \setminus X \in \mathcal{L}(\lambda) \right) \right\}.$$

Note that $\mathcal{L}(\lambda, \mu)$ is minor closed and that $\omega(\mathcal{L}(\lambda, \mu)) = \lambda + \mu + 1$. Thus a tree-decomposition over $\mathcal{L}(\lambda, \mu)$ has adhesion at most $\lambda + \mu + 1$.

Theorem 13. *Let $\mathcal{C}$ be a class of graphs with an excluded minor. Then there exist $\lambda, \mu \in \mathbb{N}$ such that all $G \in \mathcal{C}$ have a tree-decomposition over $\mathcal{L}(\lambda, \mu)$.*

Proof: This follows immediately from Theorem 12 and Proposition 11. □

For algorithmic applications we have in mind, Theorem 13 alone is not enough; we also have to compute a tree-decomposition of a given graph over $\mathcal{L}(\lambda, \mu)$. Fortunately, Robertson and Seymour have proved another deep result that helps us with this task:

Theorem 14 (Robertson and Seymour [18]). *Every minor closed class of graphs has a polynomial time membership test.*

Lemma 15. *Let $\mathcal{C}$ be a minor closed class of graphs.*

Then there is a polynomial time algorithm that computes, given a graph G, a tree-decomposition of G over $\mathcal{C}$, or rejects G if no such tree-decomposition exists.

Proof: Note that the class $\mathcal{T}$ of all graphs that have a tree-decomposition over $\mathcal{C}$ is minor closed. Thus by Theorem 14 we have polynomial time membership tests for both $\mathcal{C}$ and $\mathcal{T}$.

Without loss of generality, we can assume that $\mathcal{C}$ is not the class of all graphs. Thus the clique number $\omega := \omega(\mathcal{C})$ is finite. Recall that every tree-decomposition over $\mathcal{C}$ has adhesion at most ω. Our algorithm uses the following observation to recursively construct a tree-decomposition of the input graph G:

> *$G \in \mathcal{T}$ if, and only if, $G \in \mathcal{C}$ or there is a set $X \subseteq V^G$ such that $|X| \leq \omega$, $G \setminus X$ has at least two connected components, and for all components C of $G \setminus X$ we have $\langle X \cup C \rangle^G \cup K_X \in \mathcal{T}$.*

We omit the details. □

In particular, we are going to apply this result to the minor closed classes $\mathcal{L}(\lambda, \mu)$.

5. Approximation algorithms

Optimization problems. An *NP-optimization problem* is a tuple (I, S, C, opt), consisting of a polynomial time decidable set I of *instances*, a mapping S that associates a non-empty set $S(x)$ of *solutions* with each $x \in I$ such that the binary relation $\{(x, y) \mid y \in S(x)\}$ is polynomial time computable and there is a $k \in \mathbb{N}$ such that for all $x \in I$, $y \in S(x)$ we have $\|y\| \leq \|x\|^k$, a polynomial time computable *cost* (or *value*) function $C : \{(x, y) \mid x \in I, y \in S(x)\} \to \mathbb{N}$, and a *goal* opt $\in \{\min, \max\}$.

Given an $x \in I$, we want to find a $y \in S(x)$ such that

$$C(x, y) = \text{opt}(x) := \text{opt}\{C(x, z) \mid z \in S(x)\}.$$

Let $x \in I$ and $\epsilon > 0$. A solution $y \in S(x)$ for x is ϵ-*close* if

$$(1 - \epsilon)\text{opt}(x) \leq C(x, y) \leq (1 + \epsilon)\text{opt}(x).$$

A *polynomial time approximation scheme (PTAS)* for (I, S, C, opt) is a uniform family $(A_\epsilon)_{\epsilon>0}$ of approximation algorithms, where A_ϵ is a polynomial time algorithm that, given an $x \in I$, computes an ϵ-close solution for x in polynomial time. Uniformity means that there is an algorithm that, given ϵ, computes A_ϵ.

Note that no restrictions are made on the dependence of the runtime of the algorithms A_ϵ on ϵ. It can be very bad, and unfortunately it will be for our algorithms.

The levels of graphs of bounded local tree-width. For graph G, a vertex $v \in V^G$, and integers $j \geq i \geq 0$ we let

$$L_v^G[i, j] := \{w \in V^G \mid i \leq d^G(v, w) \leq j\}.$$

To keep the notation uniform, we are actually going to write $L_v^G[i, j]$ for arbitrary $i, j \in \mathbb{Z}$, with the understanding that $L_v^G[i, j] := \emptyset$ for $i > j$ and $L_v^G[i, j] := L_v^G[0, j]$ for $i \leq 0$.

Lemma 16. *Let $\lambda \in \mathbb{N}$. Then for all $G \in \mathcal{L}(\lambda)$, $v \in V^G$, and $i, j \in \mathbb{Z}$ with $i \leq j$ we have* $\text{tw}\big(\langle L_v^G[i, j]\rangle\big) \leq \lambda \cdot (j - i + 1)$.

Proof: First note that $L_v^G[1, j] \subseteq L_v^G[0, j] = N_j^G(v)$, thus the claim holds for $i \leq 1$. For $i \geq 2$, consider the minor H of G obtained by contracting the connected subgraph $\langle L_v^G[0, i-1]\rangle$ to a single vertex v'. Then we have $L_v^G[i, j] \subseteq N_{j-i+1}^H(v')$, and the claim follows. □

Minimum vertex cover. Instances of MINIMUM VERTEX COVER are graphs G, solutions are sets $X \subseteq V^G$ such that for every edge $vw \in E^G$ either $v \in X$ or $w \in X$ (such sets X are called *vertex covers*), the cost function is defined by $C(G, X) := |X|$, and the goal is min.

Lemma 17 (Arnborg, Lagergren, Seese [3]).
For every $k \geq 1$, the restriction of MINIMUM VERTEX COVER *to instances of tree-width at most k is solvable in linear time.*

Theorem 18. *Let $\mathcal{C}$ be a class of graphs with an excluded minor. Then the restriction of* MINIMUM VERTEX COVER *to instances in $\mathcal{C}$ has a PTAS.*

Proof: Applying Theorem 13, we choose $\lambda, \mu \in \mathbb{N}$ such that every $G \in \mathcal{C}$ has a tree-decomposition over $\mathcal{L}(\lambda, \mu)$. Let $\epsilon > 0$; we shall describe a polynomial time algorithm that, given a graph $G \in \mathcal{C}$, computes an ϵ-close solution for MINIMUM VERTEX COVER on G. Uniformity will be clear from our description. Let $k = \lceil \frac{1}{\epsilon} \rceil$ and note that $\frac{k+1}{k} \leq (1 + \epsilon)$.

In a first step, let us prove that the restriction of MINIMUM VERTEX COVER to instances in $\mathcal{L}(\lambda)$ has a PTAS.

Let $G \in \mathcal{L}(\lambda)$ and $v \in V^G$ arbitrary. For $1 \leq i \leq k$ and $j \geq 0$ we let $L_{ij} := L_v^G[(j-1)k + i, jk + i]$. By Lemma 16, $\mathrm{tw}(\langle L_{ij} \rangle) \leq \lambda(k + 1)$.

For $1 \leq i \leq k$, $j \geq 0$ let X_{ij} be a minimal vertex cover of $\langle L_{ij} \rangle$. We let $X_i := \bigcup_{j \geq 0} X_{ij}$. Then X_i is a vertex cover of G. Let X_{min} be a minimal vertex cover for G. We have $|X_{ij}| \leq |X_{\text{min}} \cap L_{ij}|$, because $X_{\text{min}} \cap L_{ij}$ is also a vertex cover of $\langle L_{ij} \rangle$. Hence

$$\sum_{i=1}^{k} |X_i| \leq \sum_{i=1}^{k} \sum_{j \geq 0} |X_{ij}| \leq \sum_{i=1}^{k} \sum_{j \geq 0} |L_{ij} \cap X_{\text{min}}| \leq (k+1)|X_{\text{min}}|.$$

The last inequality follows from the fact that every $v \in V^G$ is contained in at most $(k + 1)$ sets L_{ij}.

Choose $m, 1 \leq m \leq k$ such that $|X_m| = \min\{|X_1|, \dots, |X_k|\}$. Then

$$|X_m| \leq \frac{k+1}{k} |X_{\text{min}}| \leq (1 + \epsilon)|X_{\text{min}}|.$$

Since the X_{ij} can be computed in polynomial time by Lemma 17, X_m can also be computed in polynomial time.

In a second step, we show how to extend this approximation algorithm to classes $\mathcal{L}(\lambda, \mu)$ for $\lambda, \mu \geq 0$. Let $G \in \mathcal{L}(\lambda, \mu)$ and $U \subseteq V^G$ such that $|U| \leq \mu$ and $H := G \setminus U \in \mathcal{L}(\lambda, 0)$. The following extension of Lemma 17 can be proved by standard dynamic programming techniques (cf. [3]):

Lemma 19. *For every $k \geq 0$, the following problem can be solved in linear time: Given a graph G, a subset $U \subseteq V^G$ such that $\mathrm{tw}(G \setminus U) \leq k$, and a subset $Y \subseteq U$, compute a set $X \subseteq V^G \setminus U$ of minimal order such that $X \cup Y$ is a vertex cover of G, if such a set exists, or reject otherwise.*

For every $Y \subseteq U$ we shall compute an $X(Y) \in \mathrm{Pow}(V^G \setminus U) \cup \{\bot\}$ such that either $X(Y) \cup Y$ is a vertex cover of G and

$$|X(Y)| \leq (1 + \epsilon)\min\{|X| \mid X \subseteq V^G \setminus U, X \cup Y \text{ vertex cover of } G\},$$

or $X(Y) := \bot$ if no such $X(Y)$ exists. Using Lemma 19 instead of Lemma 17, we can do this analogously to the first step.

Then we choose a $Y_0 \subseteq U$ such that $|X(Y_0) \cup Y_0|$ is minimal. Here we define $\bot \cup Z := \bot$ for all Z and $|\bot| := \infty$. Then clearly $X(Y_0) \cup Y_0$ is an ϵ-close solution for MINIMUM VERTEX COVER on G. Moreover, since $|U| \leq \mu$, there are at most 2^μ

sets $Y \subseteq U$, so $X(Y_0) \cup Y_0$ can be computed in polynomial time (remember that μ is a constant only depending on the class $\mathcal{C}$).

In the third step, we extend our PTAS to graphs that have a tree-decomposition over $\mathcal{L}(\lambda, \mu)$, i.e. to all graphs in $\mathcal{C}$.

So let G be such a graph. We first compute a tree-decomposition $(T, (B_t)_{t \in V^T})$ of G over $\mathcal{L}(\lambda, \mu)$. Remember that by Lemma 15 this is possible in polynomial time. Recall that r^T denotes the root of T and that, for every $t \in V^T$ with parent u, we let $A_t = B_t \cap B_u$. For every $t \in V^T$, we let S_t be the subtree of T with root t, that is, the subtree with vertex set $\{s \mid t \text{ occurs on the path from } s \text{ to } r^T\}$. We let $C_t := \bigcup_{s \in S_t} B_t$.

Inductively from the leaves to the root, for every node $t \in V^T$ and for every $Y \subseteq A_t$ we compute an $X(t, Y) \in \text{Pow}(C_t \setminus A_t) \cup \{\bot\}$ such that either $X(t, Y) \cup Y$ is a vertex cover of $\langle C_t \rangle$ and

$$|X(t, Y)| \leq (1 + \epsilon)\text{min}\{|X| \mid X \cup Y \text{ vertex cover of } \langle C_t \rangle\},$$

or $X(t, Y) := \bot$ if no such vertex set exists. Since a tree-decomposition over $\mathcal{L}(\lambda, \mu)$ has adhesion at most $\lambda + \mu + 1$ we have $|A_t| \leq \lambda + \mu + 1$, thus for every $t \in V^T$ we have to compute at most $2^{\lambda + \mu + 1}$ sets $X(t, Y)$. For the root r^T we have $A_{r^T} = \emptyset$ and $\langle C_{r^T} \rangle = G$, so $X(r^T, \emptyset)$ is an ϵ-close solution for MINIMUM VERTEX COVER on G.

Suppose that $t \in V^T$ and that for every child t' of T we have already computed the family $X(t', \cdot)$. Let $U \subseteq B_t$ such that $|U| \leq \mu$ and $[B_t] \setminus U \in \mathcal{L}(\lambda)$. Let $W := U \cup A_t$ and let $Z \subseteq W$. Let $X_{\text{min}}(Z) \in \text{Pow}(C_t \setminus W) \cup \{\bot\}$ be a vertex set of minimal order such that $X_{\text{min}}(Z) \cup Z$ is a vertex cover of $\langle C_t \rangle$, or $X(Z) := \bot$ if no such vertex set exists.

We show how to compute an $X(Z) \in \text{Pow}(C_t \setminus W) \cup \{\bot\}$ such that $X(Z) \cup Z$ is a vertex cover of $\langle C_t \rangle$ and $|X(Z)| \leq (1 + \epsilon)|X_{\text{min}}(Z)|$, if $X_{\text{min}}(Z) \neq \bot$, or $X(Z) = \bot$ otherwise. Then for every $Y \subseteq A_t$ we choose a $Z \subseteq W$ such that $Y \subseteq Z$ with minimal $|X(Z) \cup (Z \setminus Y)|$ (among all $Z \supseteq Y$) and let $X(t, Y) := X(Z)$. Note that, since $|U| \leq \mu$, for every Y we have to compute at most 2^μ sets $X(Z)$ to determine $X(t, Y)$.

So let us fix a $Z \subseteq W$; we show how to compute $X(Z)$ in polynomial time.

If $W = B_t$ we let $X(Z) := \bigcup_{t' \text{ child of } t} X(t', A_{t'} \cap Z)$.

Otherwise, we choose an arbitrary $v \in B_t \setminus W$. For $1 \leq i \leq k$ and $j \geq 0$ we let $L_{ij} := L_v^{[B_t] \setminus W}[(j-1)k + i, jk + i]$. Then $\text{tw}(\langle L_{ij} \rangle) \leq \lambda(k+1)$. For $1 \leq i \leq k$ and every child t' of t there is at least one $j \geq 0$ such that $A_{t'} \setminus W \subseteq L_{ij}$, because $A_{t'}$ induces a clique in $[B_t]$. Let $j^*(i, t')$ be the least such j and $L^*_{ij} := L_{ij} \cup \bigcup_{\substack{t' \text{ child of } t \\ j^*(i,t')=j}} C_{t'} \setminus A_{t'}$.

For every $X \subseteq L_{ij}$ we let

$$X^* := X \cup \bigcup_{\substack{t' \text{ child of } t \\ j^*(i,t')=j}} X(t', (X \cup Z) \cap A_{t'})$$

We compute an $X_{ij} \subseteq L_{ij}$ with minimal $|X^*_{ij}|$ such that $X_{ij} \cup Z$ is a vertex cover of $\langle L_{ij} \cup W \rangle$ if such a vertex cover exists, and $X_{ij} = \bot$ otherwise. The usual dynamic programming techniques on graphs of bounded tree-width show that each X_{ij} can be computed in linear time if the numbers $|X(t', Y)|$ for the children t' of t are given (cf. Lemmas 17 and 19 and [3]). It is important here that every $A_{t'} \setminus W$ is a clique in $\langle L_{ij} \rangle$ and thus by Lemma 1(1) completely contained in a block of every tree-decomposition of $\langle L_{ij} \rangle$.

We let $X_i := \bigcup_{j \geq 0} X_{ij}$ and $X_i^* := \bigcup_{j \geq 0} X_{ij}^*$. Then $X_i^* \cup Z$ is a vertex cover of $\langle C_t \rangle$, if such a vertex cover exists, and $X_i = \perp$ otherwise. We choose an $i, 1 \leq i \leq k$, such that $|X_i^*| = \min\{|X_1^*|, \dots, |X_k^*|\}$ and let $X(Z) := X_i^*$. Then $X(Z)$ can be computed in polynomial time.

Recall that $X_{\text{min}} := X_{\text{min}}(Z) \subseteq C_t \setminus W$ is a vertex set of minimal order such that $X_{\text{min}} \cup Z$ is a vertex cover of $\langle C_t \rangle$, if such a vertex cover exists, and $X_{\text{min}} = \perp$ otherwise. It remains to prove that $|X(Z)| \leq (1 + \epsilon)|X_{\text{min}}|$.

Recall that for every child t' of t we have

$$|X(t', (X_{\text{min}} \cup Z) \cap A_{t'})| \leq (1 + \epsilon)|X_{\text{min}} \cap C_{t'} \setminus A_{t'}|.$$

Our construction of the X_{ij} and X_{ij}^* guarantees that for $1 \leq i \leq k, j \geq 0$ we have

$$|X_{ij}^*| \leq |X_{\text{min}} \cap L_{ij}| + \sum_{\substack{t' \text{ child of } t \\ j^*(i,t')=j}} |X(t', (X_{\text{min}} \cup Z) \cap A_{t'})|.$$

Then

$$\begin{aligned}
k|X(Z)| \leq & \sum_{i=1}^{k} |X_i^*| \\
= & \sum_{i=1}^{k} \sum_{j \geq 0} |X_{ij}^*| \\
\leq & \sum_{i=1}^{k} \sum_{j \geq 0} \Big(|X_{\text{min}} \cap L_{ij}| + \sum_{\substack{t' \text{ child of } t \\ j^*(i,t')=j}} |X(t', (X_{\text{min}} \cup Z) \cap A_{t'})| \Big) \\
\leq & \sum_{i=1}^{k} \sum_{j \geq 0} \Big(|X_{\text{min}} \cap L_{ij}| + \sum_{\substack{t' \text{ child of } t \\ j^*(i,t')=j}} (1 + \epsilon)|X_{\text{min}} \cap C_{t'} \setminus A_{t'}| \Big) \\
\leq & (k + 1)|X_{\text{min}} \cap B_t| + k(1 + \epsilon)|X_{\text{min}} \cap C_t \setminus B_t|.
\end{aligned}$$

This implies $|X(Z)| \leq (1 + \epsilon)X_{\text{min}}$. □

Minimum dominating set. Instances of MINIMUM DOMINATING SET are graphs G, solutions are sets $X \subseteq V^G$ such that for every $v \in V^G \setminus X$ there is a $w \in X$ such that $vw \in E^G$ (such sets X are called *dominating sets*), the cost function is defined by $C(G, X) := |X|$, and the goal is min.

Theorem 20. *Let $\mathcal{C}$ be a class of graphs with an excluded minor. Then the restriction of* MINIMUM DOMINATING SET *to instances in $\mathcal{C}$ has a PTAS.*

Proof: We proceed very similarly to the proof of Theorem 18, the analogous result for MINIMUM VERTEX COVER. Let $\lambda, \mu \in \mathbb{N}$ such that every graph in $\mathcal{C}$ has a tree-decomposition over $\mathcal{L}(\lambda, \mu)$. Let $\epsilon > 0$ and $k := \lceil \frac{2}{\epsilon} \rceil$.

Again, in the first step we consider the restriction of the problem to input graphs from $\mathcal{L}(\lambda)$. Given such a graph G, we choose an arbitrary $v \in V^G$. For $1 \leq i \leq k$ and $j \geq 0$ we let $L_{ij} := L_v^G[(j-1)k + i - 1, jk + i]$. Then $\text{tw}(\langle L_{ij} \rangle) \leq \lambda(k+2)$. Note that L_{ij} and $L_{i(j+1)}$ overlap in two consecutive rows, which is different from the proof of Theorem 18. The *interior* of L_{ij} is the set $L_{ij}^\circ := L_v^G[(j-1)k + i, jk + i - 1]$.

For $1 \leq i \leq k, j \geq 0$ we let $X_{ij} \subseteq L_{ij}$ be a vertex set of minimal order with the following property:

($*$) For every $w \in L_{ij}^\circ \setminus X_{ij}$ there is a $x \in X_{ij}$ such that $(w, x) \in E^G$.

Then for $1 \leq i \leq k$ the set $X_i := \bigcup_{j \geq 0} X_{ij}$ is a dominating set of G. Let m be such that $|X_m| = \min\{|X_1|, \ldots, |X_k|\}$. Computing X_m amounts to solving a variant of MINIMUM DOMINATING SET on instances of tree-width at most $\lambda(k+2)$; using the usual dynamic programming techniques, this can be done in linear time.

Since for every dominating set X of G the set $X \cap L_{ij}$ has property ($*$) we have $X_{ij} \leq X \cap L_{ij}$. Using this, we can argue as in the proof of Theorem 18 to show that X_m is an ϵ-close solution.

Adapting the second and third step of the proof of Theorem 18, it is straightforward to extend this algorithm to arbitrary input graphs in $\mathcal{C}$. □

Maximum independent set. Instances of MAXIMUM INDEPENDENT SET are graphs G, solutions are sets $X \subseteq V^G$ such that for all $v, w \in X$ we have $vw \notin E^G$ (such sets X are called *independent sets*), the cost function is defined by $C(G, X) := |X|$, and the goal is max.

Theorem 21. *Let $\mathcal{C}$ be a class of graphs with an excluded minor. Then the restriction of* MAXIMUM INDEPENDENT SET *to instances in $\mathcal{C}$ has a PTAS.*

Proof: Again we proceed similarly to the proof of Theorem 18. Let $\lambda, \mu \in \mathbb{N}$ such that every graph in $\mathcal{C}$ has a tree-decomposition over $\mathcal{L}(\lambda, \mu)$. Let $\epsilon > 0$ and $k = \lceil \frac{1}{\epsilon} \rceil$.

We describe how to treat input graphs in $\mathcal{L}(\lambda)$. Following the lines of the proof of Theorem 18, the extension to arbitrary $G \in \mathcal{C}$ is straightforward. Let $G \in \mathcal{L}(\lambda)$ and $v \in V^G$. For $1 \leq i \leq k$ and $j \geq 0$ we let $L_{ij} := L_v^G[(j-1)k + i, jk + i - 2]$. Then $\text{tw}(\langle L_{ij} \rangle) \leq \lambda(k-1)$. Note that there are no edges between L_{ij} and $L_{i(j+1)}$.

For $1 \leq i \leq k, j \geq 0$ we let X_{ij} be a maximal independent set of $\langle L_{ij} \rangle$. Then $X_i := \bigcup_{j \geq 0} X_{ij}$ is an independent set of G. Let $1 \leq m \leq k$ such that $|X_m| = \max\{|X_1|, \ldots, |X_k|\}$. Since the restriction of MAXIMUM INDEPENDENT SET to graphs of bounded tree-width is solvable in linear time, such an X_m can be computed in linear time.

Let X_{max} be a maximum independent set of G. Then for $1 \leq i \leq k, j \geq 0$ we have $|X_{ij}| \geq |X_{\text{max}} \cap L_{ij}|$. Thus

$$k|X_m| \geq \sum_{i=1}^{k} |X_i| = \sum_{i=1}^{k} \sum_{j \geq 0} |X_{ij}| \geq \sum_{i=1}^{k} \sum_{j \geq 0} |X_{\text{max}} \cap L_{ij}| \geq (k-1)|X_{\text{max}}|,$$

which implies that $X_m \geq \frac{k-1}{k}|X_{\max}| \geq (1-\epsilon)|X_{\max}|$. $\square$

Other problems. Our technique for extending approximation schemes from planar graphs to classes of graphs with excluded minors is fairly general. It is straightforward to apply it to a number of other problems that are known to have polynomial time approximation schemes on planar graphs, in particular to the other problems considered by Baker [5].

6. Other applications of Theorem 18

The tree-width of K_n-free graphs. We re-prove a theorem of Alon, Seymour, and Thomas [2] that the tree-width of a K_n-free graph G is $O(\sqrt{|G|})$. This is joint work with Reinhard Diestel and Daniela Kühn.

The technique used to prove the following lemma is the same as used in the proof of the planar separator theorem of Lipton and Tarjan [14].

Lemma 22. *Let* $\lambda \in \mathbb{N}$ *and* $G \in \mathcal{L}(\lambda)$. *Then* $\operatorname{tw}(G) \leq 3\sqrt{\lambda|G|}$.

Proof: Let $v \in V^G$ arbitrary and, for $i \geq 0$, $L_i := \{w \in V^G \mid d^G(v,w) = i\}$. Let m be maximal such that L_m is non-empty. We subdivide $\{1,\dots,m\}$ into intervals $I_1, J_1, I_2, \dots, J_{l-1}, I_l, J_l$ such that for $1 \leq i \leq l$ we have

– $|L_j| \leq \sqrt{\lambda \cdot |G|}$ for all $j \in I_i$,

– $|L_j| > \sqrt{\lambda \cdot |G|}$ for all $j \in J_i$.

For each $i \leq l, j \in I_i$ we define a path decomposition of $(P_i, (C^i_k)_{k \in V^{P_i}})$ as follows: Suppose that $I_i = [x_i, y_i]$. We let P_i be the path with vertex set $V^{P_i} := \{x_i, \dots, y_i + 1\}$ and the natural successor relation as edge relation and define the blocks by $C^i_{x_i} := L_{x_i}$, $C^i_{y_i+1} = L_{y_i}$, and $C^i_k = L_{k-1} \cup L_k$ for $x_{i+1} \leq k \leq y_i$. Since $|L_j| \leq \sqrt{\lambda \cdot |G|}$ for every $j \in I_i$, the width of this decomposition is at most $2\sqrt{\lambda \cdot |G|} - 1$. Moreover, the first block $|C^i_{x_i}|$ and the last block $|C^i_{y_i+1}|$ have size at most $\sqrt{\lambda \cdot |G|}$, and all edges out of the subgraph $\langle \bigcup_{j \in I_i} L_j \rangle$ go either from $C^i_{x_i}$ to $L_{x_i - 1}$ or from $C^i_{y_i+1}$ to L_{y_i+1}.

Since $G \in \mathcal{L}(\lambda)$ and the length of J_i is at most $\sqrt{\frac{|G|}{\lambda}}$, we have

$$\operatorname{tw}\Big(\Big\langle \bigcup_{j \in J_i} L_j \Big\rangle\Big) \leq \operatorname{ltw}^G\Big(\sqrt{\frac{|G|}{\lambda}}\Big) \leq \lambda \cdot \sqrt{\frac{|G|}{\lambda}} = \sqrt{\lambda \cdot |G|}.$$

Let $(T_i, (B^i_t)_{t \in T_i})$ be a tree-decomposition of $\langle \bigcup_{j \in J_i} L_j \rangle$ of width at most $\sqrt{\lambda \cdot |G|}$.

We combine all these decompositions to a tree-decomposition $(T, (B_t)_{t \in T})$ of G as follows: The tree T is the disjoint union of all the paths P_i and the trees T_i, connected by an edge from the first vertex x_i of P_i to an arbitrary vertex of T_{i-1} (for $2 \leq i \leq l$) and an edge from the last vertex $y_i + 1$ of P_i to an arbitrary vertex of T_i (for $1 \leq i \leq l$).

For every $i, 1 \leq i \leq l$ and $t \in V^{P_i}$ we let $B_t = C^i_t$. For every $i, 1 \leq i \leq l-1$ and $t \in V^{T_i}$ we let

$$B_t = B^i_t \cup C^i_{y_i+1} \cup C^{i+1}_{x_i},$$

that is, we add the last block of the previous path and the first block of the following path to all blocks of the tree-decomposition $(T_i, (B^i_t)_{t \in T_i})$. Moreover, for every $t \in V^{T_l}$ we let $B_t = B^l_t \cup C^l_{y_l+1}$

It is easy to verify that $(T, (B_t)_{t \in T})$ is a tree-decomposition of G of width at most $3\sqrt{\lambda|G|}$. □

Corollary 23. *Let* $\lambda, \mu \in \mathbb{N}$ *and* $G \in \mathcal{L}(\lambda, \mu)$. *Then* $\mathrm{tw}(G) \leq 3\sqrt{\lambda|G|} + \mu$.

Corollary 24 (Alon, Seymour, Thomas [2]).
Let G *be* K_n*-free. Then* $\mathrm{tw}(G) \leq O(\sqrt{|G|})$.

Deciding first-order properties. Another algorithmic application of Theorem 13 is given in [10]: It is proved that for every class $\mathcal{C}$ of graphs with an excluded minor there is a constant $c > 0$ such that for every property of graphs that is definable in first order logic there is an $O(|G|^c)$-algorithm deciding whether a given graph $G \in \mathcal{C}$ has this property. More consisely, this can be phrased as: On classes of graphs with an excluded minor, first-order model-checking is fixed-parameter tractable.

For example, this implies that for every class $\mathcal{C}$ with an excluded minor there is a constant c such that for every graph H there is an $O(|G|^c)$-algorithm testing whether a given graph $G \in \mathcal{C}$ has a subgraph isomorphic to H.

7. Further research

We have never specified the exponents and coefficients of the polynomials bounding the running times of our algorithms; they seem to be enormous. So our algorithms are only of theoretical interest. The first important step towards improving the algorithms would be a practically applicable algorithm for computing tree-decompositions of graphs of small tree-width. On the graph theoretic side, it would probably help to prove Theorem 13 directly without using Robertson's and Seymour's Theorem 12.

The traveling salesman problem is another optimization problem that has a PTAS on planar graphs [12, 4]. It would be interesting to see if this problem has a PTAS on class of graphs with an excluded minor.

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