Fixed-Point Definability and Polynomial Time

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Abstract. My talk will be a survey of recent results about the quest for a logic capturing polynomial time.

In a fundamental study of database query languages, Chandra and Harel [4] first raised the question of whether there exists a logic that captures polynomial time. Actually, Chandra and Harel phrased the question in a somewhat disguised form; the version that we use today goes back to Gurevich [15]. Briefly, but slightly imprecisely,[1] a logic L *captures* a complexity class K if exactly those properties of finite structures that are decidable in K are definable in L. The existence of a logic capturing PTIME is still wide open, and it is viewed as one of the main open problems in finite model theory and database theory. One reason the question is interesting is that we know from Fagin's Theorem [9] that existential second-order logic captures NP, and we also know that there are logics capturing most natural complexity classes above NP. Gurevich conjectured that there is no logic capturing PTIME. If this conjecture was true, this would not only imply that $\mathsf{PTIME} \neq \mathsf{NP}$, but it would also show that NP and the complexity classes above NP have a fundamentally different structure than the class PTIME and presumably most natural complexity classes below PTIME. (This aspect is highlighted by a result due to Dawar [6], also see [13].)

On the positive side, Immerman [18] and Vardi [23] proved that least fixed-point logic FP captures polynomial time on the class of all ordered finite structures. Here we say that a logic L *captures* a complexity class K *on a class* $\mathcal{C}$ of finite structures if exactly those properties of structures in $\mathcal{C}$ decidable in K are definable in L. It is easy to prove that FP does not capture PTIME on the class of all finite structures. Immerman [19] proposed the extension $\mathsf{FP}+\mathsf{C}$ of fixed-point logic by *counting operators* as a candidate for a logic capturing PTIME. It is not easy to prove, but true nevertheless, that $\mathsf{FP}+\mathsf{C}$ does not capture PTIME. This was shown by Cai, Fürer, and Immerman in 1992 [3].

Fixed-point definability on graphs with excluded minors

Even though the logic $\mathsf{FP}+\mathsf{C}$ does not capture PTIME on the class of all finite structures, it does capture PTIME on many natural classes of structures. Immerman and Lander [20] proved that $\mathsf{FP}+\mathsf{C}$ captures PTIME on the class of all trees. In 1998, I proved that $\mathsf{FP}+\mathsf{C}$ captures PTIME on the class of all planar

[1] For a precise definition of a logic capturing PTIME, I refer the reader to Grädel's excellent survey [10] on descriptive complexity theory.

graphs [11] and around the same time, Julian Mariño and I proved that $\mathsf{FP}+\mathsf{C}$ captures PTIME on all classes of structures of bounded tree width [14]. In [12], I proved the same result for the class of all K_5-free graphs, that is the class of all graphs that have no complete graph on five vertices as a minor. A *minor* of graph G is a graph H that can be obtained from a subgraph of G by contracting edges. By (the easy direction of) Kuratowski's Theorem, the class of all K_5-free graphs contains all planar graphs. We say that a class $\mathcal{C}$ of graphs *excludes a minor* if there is a graph H that is not a minor of any graph in $\mathcal{C}$. Very recently, I proved the following theorem, which generalises all these previous results, because all classes of graphs appearing in these results exclude minors.

Theorem[2] $\mathsf{FP}+\mathsf{C}$ *captures* PTIME *on all classes of graphs that exclude a minor.*

The main part of my talk will be devoted to this theorem.

Stronger logics

While $\mathsf{FP}+\mathsf{C}$ captures PTIME on many interesting classes of structures, it has been known for almost twenty years that it does not capture PTIME. So what about stronger logics? Currently, the two main candidates for logics capturing PTIME are *choiceless polynomial time with counting* $\mathsf{CP}+\mathsf{C}$ and *fixed-point logic with a rank operator* $\mathsf{FP}+\mathsf{R}$. The logic $\mathsf{CP}+\mathsf{C}$ was introduced ten years ago by Blass, Gurevich and Shelah [1] (also see [2, 8]). The formal definition of the logic is carried out in the framework of *abstract state machines*. Intuitively $\mathsf{CP}+\mathsf{C}$ may be viewed as a version of $\mathsf{FP}+\mathsf{C}$ where quantification and fixed-point operators not only range over elements of a structure, but instead over all objects that can be described by $O(\log n)$ bits, where n is the size of the structure. This intuition can be formalised in an expansion of a structure by all hereditarily finite sets which use the elements of the structure as atoms. The logic $\mathsf{FP}+\mathsf{R}$, introduced recently in [7], is an extension of FP by an operator that determines the rank of definable matrices in a structure. This may be viewed as a higher dimensional version of a counting operator. (Counting appears as a special case of diagonal $\{0,1\}$-matrices.)

Both $\mathsf{CP}+\mathsf{C}$ and $\mathsf{FP}+\mathsf{R}$ are known to be strictly more expressive than $\mathsf{FP}+\mathsf{C}$. Indeed, both logics can express the property used by Cai, Fürer, and Immerman to separate $\mathsf{FP}+\mathsf{C}$ from PTIME. For both logics it is open whether they capture polynomial time, and it is also open whether one of them semantically contains the other.

Is there a logic capturing PTIME ?

Let me close this note by a few thoughts on the question of whether there exists a logic capturing PTIME. As I said in the first paragraph, the question is still wide

[2] As this result has not been published yet, and not even a complete readable manuscript exists, the skeptic reader may treat this as a conjecture rather than a theorem.

open. By this, I do not only mean that we are far from having a proof settling the question in either direction, but actually that I do not see any evidence pointing towards one or the other answer (despite Gurevich's conjecture). If anything, I mildly lean towards believing that there is a logic for PTIME. It is known that there is a connection between the question of whether there exists a logic capturing PTIME and a variant of the graph isomorphism problem: If there is a polynomial time graph canonisation algorithm, then there is a logic capturing PTIME. But this does not really help, because the question for polynomial time graph isomorphism and canonisation algorithms is open just the same, and even if there is no polynomial time canonisation algorithm, this does not mean that there is no logic for PTIME. To gain a better understanding of the relation between the two problems, it would be interesting to see a small complexity class like uniform AC^0, which provably does not admit graph canonisation, can be captured by a logic.

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