Algorithmic Meta Theorems for Sparse Graph Classes

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Abstract. Algorithmic meta theorems give efficient algorithms for classes of algorithmic problems, instead of just individual problems. They unify families of algorithmic results obtained by similar techniques and thus exhibit the core of these techniques. The classes of problems are typically defined in terms of logic and structural graph theory. A well-known example of an algorithmic meta theorem is Courcelle's Theorem, stating that all properties of graphs of bounded tree width that are definable in monadic second-order logic are decidable in linear time.
This paper is a brief and nontechnical survey of the most important algorithmic meta theorems.

Introduction

It is often the case that a wide range of algorithmic problems can be solved by essentially the same technique. Think of dynamic programming algorithms on graphs of bounded tree width or planar graph algorithms based on layerwise (or outerplanar) decompositions. In such situations, it is natural to try to find general conditions under which an algorithmic problem can be solved by these techniques—this leads to *algorithmic meta theorems*. However, it is not always easy to describe such conditions in a way that is both mathematically precise and sufficiently general to be widely applicable. Logic gives us convenient ways of doing this. An early example of an algorithmic meta theorem based on logic is Papadimitriou and Yannakakis's [40] result that all optimisation problems in the class MAXSNP, which is defined in terms of a fragment of existential second-order logic, admit constant-ratio polynomial time approximation algorithms.

Besides logic, most algorithmic meta theorems have structural graph theory as a second important ingredient in that they refer to algorithmic problems restricted to specific graph classes. The archetypal example of such a meta theorem is Courcelle's Theorem [3], stating that all properties of graphs of bounded tree width that are definable in monadic second-order logic are decidable in linear time.

The main motivation for algorithmic meta theorems is to understand the core and the scope of certain algorithmic techniques by abstracting from problem-specific details. Sometimes meta theorems are also crucial for obtaining new algorithmic results. Two recent examples are

– a quadratic-time algorithm for a structural decomposition of graphs with excluded minors [27], which uses Courcelle's Theorem;
– a logspace algorithm for deciding wether a graph is embeddable in a fixed surface [16], which uses a logspace version of Courcelle's Theorem [14].

Furthermore, meta theorems often give a quick and easy way to see that certain problems can be solved efficiently (in principle), for example in linear time on graphs of bounded tree width. Once this has been established, a problem-specific analysis may yield better algorithms. However, an implementation of Courcelle's Theorem has shown that the direct application of meta theorems can yield competitive algorithms for common problems such as the dominating set problem [37].

In the following, I will give an overview of the most important algorithmic meta theorems, mainly for decision problems. For a more thorough introduction, I refer the reader to the surveys [26,28,33].

Meta Theorems for Monadic Second-Order Logic

Recall Courcelle's Theorem [3], stating that all properties of graphs of bounded tree width that are definable in monadic second-order logic (MSO) are decidable in linear time. Courcelle, Makowsky, and Rotics [5] generalised this result to graph classes of bounded clique width, whereas Kreutzer and Tazari showed in a series of papers [34,35,36] that polylogarithmic tree width is a necessary condition for meta theorems for MSO on graph classes satisfying certain closure conditions like being closed under taking subgraphs (also see [23]).

In a different direction, Elberfeld, Jakoby, and Tantau [14,15] proved that all MSO-definable properties of graphs of bounded tree width can be decided in logarithmic space and all MSO-definable properties of graphs of bounded tree depth can be decided in AC^0.

Meta Theorems for First-Order Logic

For properties definable in first-order logic (FO), we know meta theorems on a much larger variety of graph classes. Before I describe them, let me point out an important difference between FO-definable properties and MSO-definable properties. In MSO, we can define NP-complete problems like 3-COLOURABILITY. Thus if we can decide MSO-definable properties of a certain class of graphs in polynomial time, this is worth noting. But the range of graph classes where we can hope to achieve this is limited. For example, the MSO-definable property 3-COLOURABILITY is already NP-complete on the class of planar graphs. On the other hand, all FO-definable properties of (arbitrary) graphs are decidable in polynomial time (even in uniform AC^0). When proving meta theorems for FO, we are usually interested in linear-time algorithms or polynomial-time algorithms with a small fixed exponent in the running time, that is, in fixed-parameter tractability (see, for example, [10,19]). When we say that NP-complete problems like MINIMUM DOMINATING SET are definable in FO, we mean that for each k there is an FO-sentence ϕ_k stating that a graph has a dominating set of size at most k. Thus we define (the decision version of) the DOMINATING SET problem by a family of FO-sentences, and if we prove that FO-definable properties can be decided in linear time on a certain class of graphs, this implies that DOMINATING SET parameterized by the size of the solution is fixed-parameter tractable on this class of graphs. By comparison, we can define 3-COLOURABILITY by a single MSO-sentence, and if we prove that MSO-definable properties can be decided in linear time on a certain class of graphs, this implies that 3-COLOURABILITY can be decided in linear time on this class of graphs.

After this digression, let us turn to the results. The first notable meta theorem for deciding FO-definable properties, due to Seese [41], says that FO-definable properties of bounded-degree graphs can be decided in linear time. Frick and Grohe [21] gave linear-time algorithms for deciding FO-definable properties of planar graphs and all apex-minor-free graph classes and $O(n^{1+\epsilon})$ algorithms for graph classes of bounded local tree width. Flum and Grohe [18] proved that deciding FO-definable properties is fixed-parameter tractable on graph classes with excluded minors, and Dawar, Grohe, and Kreutzer [8] extended this to classes of graphs locally excluding a minor. Dvořák, Král, and Thomas [13] proved that FO-definable properties can be decided in linear time on graph classes of bounded expansion and in time $O(n^{1+\epsilon})$ on classes of locally bounded expansion. Finally, Grohe, Kreutzer, and Siebertz [30] proved that FO-definable properties can be decided in linear time on nowhere dense graph classes. Figure 1 shows the containment relation between all these and other sparse graph classes. Nowhere dense classes were introduced by Nešetřil and Ossona de Mendez [38,39] (also see [29]) as a formalisation of classes of "sparse" graphs. They include most familiar examples of sparse graph classes like graphs of bounded degree and planar graphs. Notably, classes of bounded average degree or bounded degeneracy are not necessarily nowhere dense.

The meta theorem for FO-definable properties of nowhere dense classes is optimal if we restrict our attention to graph classes closed under taking subgraphs: if $\mathcal{C}$ is a class of graphs closed under taking subgraphs that is somewhere dense (that is, not nowhere dense), then deciding FO-properties of graphs in $\mathcal{C}$ is as hard as deciding FO-properties of arbitrary graphs, with respect to a suitable form of reduction [13,33]. Thus under the widely believed complexity-theoretic assumption $\mathsf{FPT} \neq \mathsf{AW}[*]$, which is implied by more familiar assumptions like the exponential time hypothesis or $\mathsf{FPT} \neq \mathsf{W}[1]$, deciding FO-definable properties of graphs from $\mathcal{C}$ is not fixed-parameter tractable.

There are a few meta theorems for FO-definable properties of graph classes that are somewhere dense (and hence not closed under taking subgraphs). Ganian et al. [24] give quasilinear-time algorithms for certain classes of interval graphs. By combining the techniques of [5] and [21], it can easily be shown that deciding FO-definable properties is fixed-parameter tractable on graphs of

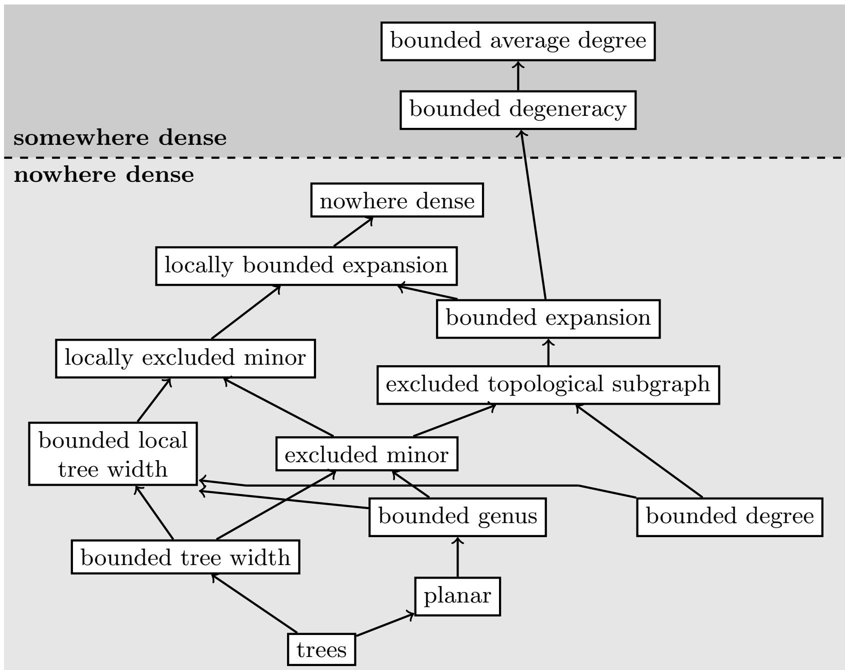

Fig. 1. Sparse graph classes

bounded local rank width (see [26]). It is also easy to prove fixed-parameter tractability for classes of unbounded, but slowly growing degree [11,25].

Counting, Enumeration, and Optimisation Problems

Many of the meta theorems above have variants for counting and enumeration problems, where we are given a formula $\phi(x_1, \dots, x_k)$ with free variables and want to compute the number of tuples satisfying the formula in a given graph or compute a list of all such tuples, and also for optimisation problems. (See, for example, [1,2,4,6,7,9,12,11,17,20,31,32].)

Uniformity

In this paper, we stated all meta theorems in the form: for every property definable in a logic L on a class $\mathcal{C}$ of graphs there is an $O(n^c)$ algorithm. Here n is the number of vertices of the input graph, and the exponent c is a small constant, most often 1 or $(1+\epsilon)$. However, all these theorems hold in a uniform version of the form: there is an algorithm that, given an L-sentence ϕ and a graph $G \in \mathcal{C}$, decides whether ϕ holds in G in time $f(k) \cdot n^c$, where k is the length of the sentence ϕ and f is some computable function (the exponent c remains the same).

For families of classes constrained by an integer parameter ℓ, such as the classes of all graphs of tree width at most ℓ or the classes of all graphs that exclude an ℓ-vertex graph as a minor, we even have an algorithm running in time $g(k, \ell) \cdot n^c$, for a computable function g.

We have seen that the exponent c in a running time $f(k) \cdot n^c$ is usually close to optimal ($c = 1$ or $c = 1 + \epsilon$ for every $\epsilon > 0$). However, the "constant factor" $f(k)$ is prohibitively large: if $\mathsf{P} \neq \mathsf{NP}$, then for every algorithm deciding MSO-definable properties on the class of trees in time $f(k) \cdot n^c$ for a fixed exponent c, the function f is nonelementary. The same holds for FO-definable properties under the stronger assumption $\mathsf{FPT} \neq \mathsf{AW}[*]$ [22]. This implies corresponding lower bounds for almost all classes considered in this paper, because they contain the class of trees (see Figure 1). An elementary dependence on k can only be achieved on classes of bounded degree [22].

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