Bounded–arity hierarchies in fixed–point logics

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Abstract. In this paper we prove that for each k, the expressive power of k–ary fixed–point logic, i.e. the fragment of fixed–point logic whose fixed–point operators are restricted to arity $\leq k$, strictly exceeds the power of $(k-1)$–ary fixed–point logic. This solves a problem that was posed by Chandra and Harel in 1982.
Our proof has a rather general form that applies to several variants of fixed–point logic and also to transitive–closure logic.

1 Introduction

Fixed–point logic is obtained by augmenting first–order logic by a *fixed–point operator*. If we bound the arity of this operator, we obtain a hierarchy inside of fixed–point logic. We are concerned with the question whether this hierarchy is strict.

There are several different fixed–point logics that are considered in finite model theory. To be as general as possible we work with Abiteboul's and Vianu's [AV88] *partially defined fixed–point logic* **PFP** which is the most expressive of the familiar fixed–point logics. $\mathbf{PFP}^k$ denotes the k–ary fragment of **PFP**.

Main theorem : *The logics* $\mathbf{PFP}^k$ $(k \geq 1)$ *form a strict hierarchy.*

As a corollary we get the analogon for other fixed–point logics and transitive closure logic.

Our proof makes use of the straightforward idea to generalize the result that connectivity of graphs is not expressible in first–order logic (which we could consider as $\mathbf{PFP}^0$), but in $\mathbf{PFP}^1$, to the proposition, that connectivity of graphs on k–tuples (of distinct elements) is not expressible in $\mathbf{PFP}^{k-1}$, but in $\mathbf{PFP}^k$.

The crucial point is that we can find connected and disconnected graphs on k–tuples which are both very homogeneous in the $(k-1)$–tuples. To find such structures we use a result of Oberschelp [Obe82] on 0–1 laws which involves a probabilistic argument (so we do not give the structures explicitly). That spares us getting lost in combinatorial difficulties.

For the non-expressibility part we introduce a game to prove that a fixed–point sentence holds in a structure; by playing it in an Ehrenfeucht-Fraïssé manner we get the result.

Since fixed–point logic plays an important role in finite model theory, it should be notified that our proof only uses finite structures. In fact, the result might be easier to prove in the scope of infinite structures because there the combinatorial side is often easier to handle.

The question whether the bounded arity hierarchy for least fixed–point logic is strict was posed by Chandra and Harel [CH82] in 1982. There it was announced that Gaifman solved the problem for small arities (≤ 3). Another partial result (for sentences with only one fixed–point operator in the beginning) was given by Dublish and Maheshwari [DM89]. The analogous problem for transitive closure logic was posed by Calo and Makowsky in [CM91], where they could prove $TC^1 \neq TC^2$. Afrati and Cosmadakis [AC89] solved the hierarchy problem for Datalog.

For the basic notions of first order logic we refer to [EFT84].
A difference to the standard definitions presented there is that we allow first–order formulae to have free second–order variables. So an interpretation $\mathfrak{I}$ is a pair $(\mathfrak{A}, \beta)$ consisting of a structure $\mathfrak{A}$ and an assignment β which is defined on the first and second order variables. We extend β to all terms.
The universe of a structure $\mathfrak{A}$ is always denoted by A. We write $\overset{k}{x}$ for the k–tuple $(x_1, \ldots, x_k)$.

2 Fixed–point logics

Partially defined fixed–point logic

2.1 Definition : (1) The class *PFP* of partially defined fixed–point formulae is given by means of the calculus consisting of the first–order rules and the rule:

$(PFP) \quad \dfrac{\varphi}{[PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}}$ where $l \geq 1$, $\overset{l}{u}$ is a tuple of terms, X is l-ary

$[PFP_{\overset{l}{x},X} \ldots](\ldots)$ is called an l–ary fixed–point operator.

(2) For $k \geq 1$, the class PFP^k of *k-ary partially defined fixed–point formulae* is the subclass of *PFP* consisting of those formulae that contain at most k-ary fixed–point operators, i.e. the (PFP)-rule is only allowed for $l \leq k$.

To define the semantics we consider an interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ and a fixed–point formula $[PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$. We define a sequence $(X_i^{\mathfrak{I}})_{i \geq 0}$ of subsets of A^l by

$$X_0^{\mathfrak{I}} := \emptyset$$
$$X_{i+1}^{\mathfrak{I}} := \{\overset{l}{a} \in A^l \mid \mathfrak{I} \models \varphi[\overset{l}{a}, X_i^{\mathfrak{I}}]\}$$
$$X_\infty^{\mathfrak{I}} := \begin{cases} X_m & \text{where } m = min\{i \mid X_i = X_{i+1}\} \text{ (if it exists)} \\ \emptyset & \text{otherwise} \end{cases}$$

We let

$$\mathfrak{I} \models [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u} \quad\Longleftrightarrow\quad \beta(\overset{l}{u}) \in X^{\mathfrak{I}}_{\infty}$$

and define the semantics of the logic **PFP** inductively.

2.2 Proviso : To avoid some unnecessary complications from now on we assume (if nothing else is said) without loss of generality that the following syntactical conditions hold for the fixed–point formulae we consider (otherwise it is easy to find logically equivalent formulae where they hold):

– Negation symbols only occur in front of atomic subformulae or in front of fixed–point operators.
– Each variable is quantified at most once in each formula (by $\exists$ or $\forall$ quantifiers or fixed–point operators). Free variables are never quantified.
– If $[PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$ is a formula with a subformula $X\ \overset{l}{t}$, then $\{\overset{l}{x}\} \cap free(\overset{l}{t}) = \emptyset$.
  (To achieve this simply replace $X\ \overset{l}{t}$ by $\exists\ \overset{l}{z}\ (\overset{l}{t} = \overset{l}{z} \wedge X\ \overset{l}{z})$, where $\overset{l}{z}$ are new variables.)

A game to evaluate fixed–point formulae

Here we give a method to prove that a certain fixed–point sentence χ holds in a structure $\mathfrak{A}$. We do this in form of a game (that we call the $(\chi, \mathfrak{A})$-*game*) between two players $\exists$ and $\forall$. $\exists$ wants to prove that χ holds in $\mathfrak{A}$ and $\forall$ that it does not.
The ideas in this section are by no means new, but based on the standard game–semantics for infinitary logic. A similar game as ours was introduced in [McC93] for least fixed–point logic.

First we want to recall the well known fact that a fixed–point operator can be treated as an infinite disjunction:
Consider the formula $\psi = [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$.
We let $\psi_0 := \neg x_1 = x_1$ and for each $i \geq 0$ ψ_{i+1} be the formula obtained from φ by replacing each subformula of the form $X\ \overset{l}{t}$ by $\exists\ \overset{l}{x}\ (\overset{l}{x} = \overset{l}{t} \wedge \psi_i)$.
Then clearly ψ_i defines the set $X_i^{\mathfrak{I}}$ for each interpretation $\mathfrak{I}$, hence we have:

$$\models \psi \leftrightarrow \bigvee_{i \geq 1} \underbrace{\big(\exists\ \overset{l}{x}\ (\overset{l}{x} = \overset{l}{u} \wedge \psi_i) \wedge \forall\ \overset{l}{x}\ (\psi_{i-1} \leftrightarrow \psi_i)\big)}_{=:\ \psi_i^*(\overset{l}{u})}$$

Note that the free variables of ψ and $\psi_i^*(\overset{l}{u})$ $(i \geq 1)$ are the same.

We should remark here that the formulae ψ_i do not have the syntactical form given by Proviso 2.2 anymore because variables are requantified. The proviso is only meant to hold for the fixed–point formulae we start with.

2.3 Definition : Let χ be a *PFP*–formula and $\mathfrak{A}$ a structure of the same signature. The $(\chi, \mathfrak{A})$*–game* is played by two players $\exists$ and $\forall$. Each situation of the game consists of a *current formula* ψ and a *current interpretation* of the free variables of ψ.
In the beginning, the current formula is χ. $\forall$ selects an interpretation of its free variables.
The game ends if ψ is atomic or negated atomic. $\exists$ wins the game if ψ holds in $\mathfrak{A}$ under the current interpretation of its variables.
Otherwise one of the following moves is made depending on the form of ψ:

$\vee$–**move:** (if $\psi = \psi_1 \vee \psi_2$)
$\exists$ selects a ψ_i $(i = 1, 2)$ to be the current formula in the next situation. The current interpretation is restricted to the free variables of ψ_i.

$\wedge$–**move:** (if $\psi = \psi_1 \wedge \psi_2$)
$\forall$ selects a ψ_i $(i = 1, 2)$ to be the current formula in the next situation. The current interpretation is restricted to the free variables of ψ_i.

$\exists$–**move:** (if $\psi = \exists x \varphi$)
The current formula in the next situation is φ. $\exists$ extends the current interpretation by an interpretation of x in $\mathfrak{A}$.

$\forall$–**move:** (if $\psi = \forall x \varphi$)
The current formula in the next situation is φ. $\forall$ extends the current interpretation by an interpretation of x in $\mathfrak{A}$.

PFP–**move:** (if $\psi = [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$)
$\exists$ selects an $i \geq 0$. The current formula in the next situation is $\psi_i^*(\overset{l}{u})$. The current interpretation remains unchanged.

$\neg$*PFP*–**move:** (if $\psi = \neg\ [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$)
$\forall$ selects an $i \geq 0$. The current formula in the next situation is the formula obtained from $\neg\psi_i^*(\overset{l}{u})$ by pushing the negation symbol inside the formula in the canonical way. The current interpretation remains unchanged.

The following theorem can be proved by an easy induction along the lines of the definition of a formula χ:

2.4 Theorem : *Let χ be a fixed–point sentence and $\mathfrak{A}$ a structure. Then the following statements are equivalent:*

(1) Player $\exists$ has a winning strategy for the $(\chi, \mathfrak{A})$-game.
(2) $\mathfrak{A} \models \chi$.

To apply this theorem later we have to analyse certain situations that may occur in the $(\chi, \mathfrak{A})$–game a bit more precisely.
First consider a situation that follows a *PFP* or $\neg$*PFP*–move (say, corresponding to the formula $\psi = (\neg)\ [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$).

This situation has a current formula $(\neg)\psi_i^*(\overset{l}{u})$. (We are a bit sloppy with the negation symbols here and will be for the rest of this paper in the same fashion: By $\neg\psi_i^*(\overset{l}{u})$ we mean the formula that is constructed from $\neg\psi_i^*(\overset{l}{u})$ in the natural way with negation symbols only occurring in front of atomic subformulae or fixed–point operators.)
We call such a situation an *F–situation*.

All subformulae of $\psi_i^*(\overset{l}{u})$ of the form $(\neg)\exists\,\overset{l}{x}\,(\overset{l}{x}{=}\overset{l}{t} \wedge \psi_j)$ that were obtained from replacing subformulae $(\neg)X\ \overset{l}{t}$ of φ are called *R–formulae*; the situations they represent are called *R–situations*. The F–situation defined above (with current formula $(\neg)\psi_i^*(\overset{l}{u})$) is called their corresponding F-situation.

The following observation will be needed later:

2.5 Observation : *Consider an R-situation R in the $(\chi,\mathfrak{A})$–game with current formula $\xi = (\neg)\exists\,\overset{l}{x}\,(\overset{l}{x}{=}\overset{l}{t} \wedge \psi_j)$ and let F be the corresponding F–situation with current formula $\nu = (\neg)\psi_i^*(\overset{l}{u})$. Then*

$$\mathit{free}(\xi) \subseteq \mathit{free}(\nu) \cup \mathit{free}(\overset{l}{t})\ .$$

Moreover, the current interpretation of all variables in $\mathit{free}(\xi) \cap \mathit{free}(\nu)$ is the same in the situations R and F.

Proof: The first statement is a consequence of the above remark that the free variables of $\psi =\ [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}$ and $\psi_i^*(\overset{l}{u})$ are the same.
The second is an immediate consequence from the fact that each variable is only quantified once in each fixed–point formula. □

We also need a notion of rank for our game. First we define the quantifier–rank $qr(\chi)$ of a fixed–point formula χ inductively, letting

$$qr(\ [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}) = qr(\varphi) + 2l\ .$$

Now we define the $\mathit{rank}(S)$ of a situation S in the $(\chi,\mathfrak{A})$–game inductively as follows:

- If S is the starting situation we let $\mathit{rank}(S) = qr(\chi)$.
- If S is not and R–situation and the previous situation was S' we let:

$$\mathit{rank}(S) = \begin{cases} \mathit{rank}(S') & \text{if the last move was a } \vee \text{ or } \wedge\text{–move} \\ \mathit{rank}(S') - 1 & \text{if the last move was an } \exists \text{ or } \forall\text{–move} \end{cases}$$

- If S is an R–situation and S' is the corresponding F–situation we let $\mathit{rank}(S) = \mathit{rank}(S') - l$.

An easy induction shows:

2.6 Observation : *For each situation S in the $(\chi, \mathfrak{A})$–game we have $rank(S) \geq 0$. Moreover, in each situation of the game the current formula has $\leq qr(\chi)$ free variables.*

Note that this would be wrong if we had defined

$$qr(\ [PFP_{\overset{l}{x},X}\ \varphi]\ \overset{l}{u}) = qr(\varphi) + l \quad \text{(instead of } +2l).$$

Other logics

Transitive closure logic **TC** (cf. [Imm87]) is obtained by augmenting the syntax of first–order logic by a *transitive closure operator*, i.e. for each formula $\varphi(\overline{x}, \overline{y})$ with two tuples $\overline{x}, \overline{y}$ of free variables of the same length we build a new formula

$$[TC_{\overline{x},\overline{y}}\ \varphi(\overline{x}, \overline{y})]\overline{x}, \overline{y}\ .$$

If we consider $\varphi(\overline{x}, \overline{y})$ as a binary relation on tuples the new formula defines the reflexive transitive closure of this relation.
If we restrict the length of the tuples $\overline{x}, \overline{y}$ to be $\leq k$, we obtain *k-ary transitive closure logic* $\mathbf{TC}^k$. Obviously we have

$$[TC_{\overset{k}{x},\overset{k}{y}}\ \varphi(\overset{k}{x}, \overset{k}{y})]\ \overset{k}{u}, \overset{k}{v} \quad \dashv\models \quad [PFP_{\overset{k}{y},X}\ \ \overset{k}{y}{=}\overset{k}{u} \vee \exists\ \overset{k}{x}\ (X\ \overset{k}{x} \wedge \varphi(\overset{k}{x}, \overset{k}{y}))]\ \overset{k}{v}$$

hence $\mathbf{TC}^k \subseteq \mathbf{PFP}^k$.
The proof of our main theorem will imply $\mathbf{TC}^{k+1} \not\subseteq \mathbf{PFP}^k$ for each $k \geq 1$, so the logics $(\mathbf{TC}^k)_{k \geq 1}$ form a strict hierarchy.

The hierarchy theorem also holds for other well known fixed–point logics such as the existential fragment of fixed–point logic **EFP** (cf.[BG87]), least fixed–point logic **LFP** (cf.[CH82]) and inductive fixed–point logic **IFP**(cf.[Gur84]).
Since the proof of our main theorem will also imply $\mathbf{EFP}^{k+1} \not\subseteq \mathbf{PFP}^k$ for each $k \geq 1$ this follows from the fact

$$\mathbf{EFP}^k \subseteq \mathbf{LFP}^k \subseteq \mathbf{IFP}^k \subseteq \mathbf{PFP}^k\ .$$

3 Extension axioms for parametric classes

In this section we take care of the probabilistic part of our proof. We want to find structures where the k–tuples (of distinct elements) can be divided into two parts, but where each $k-1$–tuple can be extended to k–tuples in both parts, i.e. structures that are somehow "homogeneous in the $(k-1)$-tuples".
The idea is to take structures with only one k-ary relation P that is chosen randomly on the set of k–tuples of distinct elements. It is plausible that if the structures are big enough the statement above holds.

To realize these ideas we make use of a theorem by Oberschelp [Obe82] concerning 0–1 laws on parametric classes.

Let σ be a vocabulary that only consists of finitely many relation symbols.

For each $n \geq 1$ we can define a probability measure l_n on the class of all σ-structures, letting

$$l_n(\mathcal{C}) = \frac{|\{\mathfrak{A} \in \mathcal{C} \mid A = \{1, \ \ldots \ , n\}\}|}{|\{\mathfrak{A} \mid \mathfrak{A} \ \sigma\text{-structure}, \ A = \{1, \ \ldots \ , n\}\}|}$$

for each class $\mathcal{C}$ of σ-structures.
l_n gives the probability that a randomly chosen σ-structure with universe $\{1, \ \ldots \ , n\}$ is in $\mathcal{C}$.

If $\mathcal{D}$ is another class of σ-structures with $l_n(\mathcal{D}) > 0$ we can define the conditional probability of $\mathcal{C}$ with respect to $\mathcal{D}$ by

$$l_n(\mathcal{C}|\mathcal{D}) = \frac{l_n(\mathcal{C} \cap \mathcal{D})}{l_n(\mathcal{D})}$$

If $\mathcal{C} = Mod(\varphi)$ we write $l_n(\varphi)$ instead of $l_n(\mathcal{C})$ and $l_n(\varphi|\mathcal{D})$ instead of $l_n(\mathcal{C}|\mathcal{D})$.

It is a well known result that a *0–1 law* holds for first–order logic, i.e. for each first–order sentence φ the *labeled asymptotic probability* $l(\varphi) = \lim_{n\to\infty} l_n(\varphi)$ exists and is 0 or 1.

Oberschelp extended this result to so called parametric classes:

3.1 Theorem ([Obe82]) : *A first–order 0-1 law holds on parametric classes, i.e. if $\mathcal{P}$ is a parametric class then for each first–order sentence φ the conditional labeled asymptotic probability $l(\varphi|\mathcal{P}) = \lim_{n\to\infty} l_n(\varphi|\mathcal{P})$ exists and is 0 or 1.*

We do not want to introduce parametric classes formally (they are classes that are definable by certain universal first–order sentences) but just point out that for each $k \geq 1$ the class $\mathcal{P}_k$ of $\{P\}$–structures with a k–ary relation P that only contains k–tuples of distinct elements is parametric.
Oberschelps theorem is proved (cf. [Com88]) by making use of the so called *extension axioms*:

3.2 Definition : (1) A first–order formula $\varphi(\overset{r}{v})$ is called an *atomic r-type* if there exists a structure $\mathfrak{A}$ and a tuple $\overset{r}{a} \in A$ such that $\varphi = \varphi^0_{\mathfrak{A},\overset{r}{a}}$, where

$$\varphi^0_{\mathfrak{A},\overset{r}{a}} := \bigwedge \Big\{\theta \mid \theta \text{ atomic or negated atomic first–order}, \\ free(\theta) \subseteq \{v_1, \ldots, v_r\}, \mathfrak{A} \models \theta[\overset{r}{a}]\Big\}$$

(2) Let $\varphi(\overset{r}{v})$, $\psi(\overset{r+1}{v})$ be atomic types such that $\psi(\overset{r+1}{v})$ extends $\varphi(\overset{r}{v})$,
i.e. $\models \psi(\overset{r+1}{v}) \to \varphi(\overset{r}{v})$.

Then we call the sentence $\forall v_1 \dots \forall v_r(\varphi(\overset{r}{v}) \rightarrow \exists v_{r+1}\psi(\overset{r+1}{v}))$ an $(r+1)$-extension axiom.

Φ_r denotes the set of all r–extension axioms.

(3) Let $\mathcal{P}$ be a parametric class. We call an extension axiom $\forall v_1 \dots \forall v_r(\varphi(\overset{r}{v}) \rightarrow \exists v_{r+1}\psi(\overset{r+1}{v}))$ compatible with $\mathcal{P}$ if $\exists v_1 \dots \exists v_{r+1}\psi$ has a model in $\mathcal{P}$.

$\Phi_r(\mathcal{P})$ denotes the set of all r-extension axioms compatible with $\mathcal{P}$.

It can be shown, that $l(\bigwedge \Phi_r(\mathcal{P})|\mathcal{P}) = 1$ for each parametric class $\mathcal{P}$ and $r \geq 1$. Furthermore, for each first–order sentence φ of quantifier rank $\leq r$ either $\Phi_r(\mathcal{P}) \models \varphi$ or $\Phi_r(\mathcal{P}) \models \neg\varphi$.
That proves Theorem 3.1.

Turning to the parametric class $\mathcal{P}_k$ defined above this implies that there exists an m depending on k and r such that $l_m(\bigwedge \Phi_r(\mathcal{P}_k)|\mathcal{P}_k) > \frac{1}{2}$ and we immediately get:

3.3 Lemma : *Let $k, r \geq 1$, P k-ary. Then we can find an $m(k,r) \geq 1$ and $\{P\}$-structures $\mathfrak{L}(k,r), \mathfrak{R}(k,r) \models \Phi_r(\mathcal{P}_k)$ with universe $\{1, \dots, m(k,r)\}$ such that for all distinct $a_1, \dots, a_k \in \{1, \dots, m(k,r)\}$:*

$$(\overset{k}{a} \in P^{\mathfrak{L}(k,r)} \iff \overset{k}{a} \notin P^{\mathfrak{R}(k,r)})$$

Proof: For each $\mathfrak{A} \models \Phi_r(\mathcal{P}_k)$ with universe $\{1, \dots, m\}$ there exists exactly one $\mathfrak{B}_{\mathfrak{A}} \models \mathcal{P}_k$ with universe $\{1, \dots, m\}$ such that for all distinct $a_1, \dots, a_k \in \{1, \dots, m\}$:

$$(\overset{k}{a} \in P^{\mathfrak{A}} \iff \overset{k}{a} \notin P^{\mathfrak{B}_{\mathfrak{A}}})$$

But if $\mathfrak{B}_{\mathfrak{A}} \not\models \Phi_r(\mathcal{P}_k)$ for all $\mathfrak{A}$, we have $l_m(\bigwedge \Phi_r(\mathcal{P}_k)|\mathcal{P}_k) \leq \frac{1}{2}$. □

$\mathfrak{L}(k,r)$ and $\mathfrak{R}(k,r)$ are the structures we aimed at in the introduction of this section.

4 Proof of the main theorem

Throughout this section we let $k > 1$ be a fixed integer and $\tau = \{E, c_1, \dots, c_k, d_1, \dots, d_k\}$ a vocabulary, where E is a $2k$-ary relation symbol and $c_1, \dots, c_k, d_1, \dots, d_k$ are constant symbols.

We consider (certain) τ-structures as directed graphs on k–tuples with two distinguished points $\overset{k}{c}$ and $\overset{k}{d}$.
Then the query

THERE IS A PATH FROM $\overset{k}{c}$ TO $\overset{k}{d}$.

is expressible by the k-ary fixed–point formula

$$\chi := [PFP_{\overset{k}{y},X}\ \overset{k}{y}=\overset{k}{c} \vee \exists\ \overset{k}{x}\ (X\ \overset{k}{x} \wedge E\ \overset{k}{x}\overset{k}{y})]\ \overset{k}{d}\ ,$$

—which is in fact in EFP^k — or by the k–ary transitive closure formula

$$[TC_{\overset{k}{x},\overset{k}{y}}\ E\ \overset{k}{x}\overset{k}{y}]\ \overset{k}{c}, \overset{k}{d}\ .$$

4.1 Proposition : *There exists no PFP^{k-1} formula that is equivalent to χ.*

Giving an indirect proof we suppose that χ^* was such a formula. Let $q := qr(\chi^*) + k$.

We first construct structures $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} \models \chi$ and $\mathfrak{B} \not\models \chi$ and prove that player $\exists$ wins the $(\chi^*, \mathfrak{B})$-game afterwards. By Theorem 2.4 this leads to a contradiction.

The structures

Depending on parameters k, q, n we define structures $\mathfrak{A} = \mathfrak{A}(k, q, n)$ and $\mathfrak{B} = \mathfrak{B}(k, q, n)$.

Intuitively we proceed as follows:
We take n disjoint copies of the structure $\mathfrak{L}(k, q)$ (or $\mathfrak{R}(k, q)$, that depends only on the viewpoint), which we call the *rows* of our new structures. We order these rows and define the relation E to hold for each pair $(\overset{k}{a}, \overset{k}{b})$ of tuples of distinct elements in succeeding rows, where either both $\overset{k}{a}$ and $\overset{k}{b}$ are in $P^{\mathfrak{L}(k,q)}$ (in their respective rows) or both are in $P^{\mathfrak{R}(k,q)}$. Now we erase the relation P in each row and obtain an E-structure that can be considered as a graph on k–tuples having two connected components.

We get our τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ by letting both $\overset{k}{c}$ and $\overset{k}{d}$ be in the $P^{\mathfrak{L}(k,q)}$ component in $\mathfrak{A}$ and letting $\overset{k}{c}$ be in the $P^{\mathfrak{L}(k,q)}$ component and $\overset{k}{d}$ in the $P^{\mathfrak{R}(k,q)}$ component in $\mathfrak{B}$.

4.2 Definition : For all $k, q, n \geq 1$ we define structures $\mathfrak{A} = \mathfrak{A}(k, q, n)$ and $\mathfrak{B} = \mathfrak{B}(k, q, n)$ as follows:

$A := B := \{1, \ldots, n\} \times \{1, \ldots, m\}$ (where $m = m(k, q)$),

$$E^{\mathfrak{A}} := E^{\mathfrak{B}} := \Big\{ \big((I, a_1), \ldots, (I, a_k), (I+1, b_1), \ldots, (I+1, b_k)\big) \mid 1 \leq I < n,$$

$$\forall i, j \leq k : 1 \leq a_i \neq a_j, b_i \neq b_j \leq m,\ (\overset{k}{a} \in P^{\mathfrak{L}(k,q)} \Leftrightarrow \overset{k}{b} \in P^{\mathfrak{L}(k,q)}) \Big\}$$

To define $\overset{k}{c}$ and $\overset{k}{d}$ we select tuples $\overset{k}{a} \in P^{\mathfrak{L}(k,q)}$ and $\overset{k}{b} \in P^{\mathfrak{R}(k,q)}$ and let for all $i \le k$:

$$c_i^{\mathfrak{A}} := c_i^{\mathfrak{B}} := (1, a_i) \,, \quad d_i^{\mathfrak{A}} := (n, a_i) \quad \text{and} \quad d_i^{\mathfrak{B}} := (n, b_i)$$

We call the set $\{(I, a) \mid a \le m\}$ the Ith row of $\mathfrak{A}$ or $\mathfrak{B}$ $(1 \le I \le n)$.

Obviously we have $\mathfrak{A} \models \chi$ and $\mathfrak{B} \not\models \chi$.
The next lemma shows that we can treat the rows of our structures independently.

4.3 Lemma : *Let $k, q \ge 1, n \ge 2$, $\overset{s_0}{a_0}, \overset{s_0}{b_0}, \overset{s_1}{a_1}, \overset{s_1}{b_1} \in \{1, \ldots, m(k,q)\}$ such that the two mappings defined by $a_{0j} \mapsto b_{0j}$ $(1 \le j \le s_0)$ and $a_{1j} \mapsto b_{1j}$ $(1 \le j \le s_1)$ are partial isomorphisms from $\mathfrak{L}(k,q)$ to $\mathfrak{L}(k,q)$ (case 1) or from $\mathfrak{L}(k,q)$ to $\mathfrak{R}(k,q)$ (case 2).*

In both cases for all $I \in \{2, \ldots, n-2\}$ the mapping defined by

$$(I, a_{0j}) \mapsto (I, b_{0j}) \quad (1 \le j \le s_0)$$

$$(I+1, a_{1j}) \mapsto (I+1, b_{1j}) \quad (1 \le j \le s_1)$$

is a partial isomorphism from $\mathfrak{A}(k,q,n)$ to $\mathfrak{B}(k,q,n)$.

This also holds for the first and last row ($I = 1$ or $I + 1 = n$ respectively) if we take care of the constants $\overset{k}{c}$ and $\overset{k}{d}$ in the obvious way. (But note that for $I = 1$ only case 1 and for $I + 1 = n$ only case 2 may occur because of the definition of the constants $\overset{k}{c}$ and $\overset{k}{d}$ in structures $\mathfrak{A}(k,q,n)$ and $\mathfrak{B}(k,q,n)$.)

Proof: Take for example case 2:
Clearly the mapping is injective, and for tuples $\overset{k}{a} \in \{\overset{s_0}{a_0}\}$, $\overset{k}{a'} \in \{\overset{s_1}{a_1}\}$ of distinct elements corresponding to $\overset{k}{b} \in \{\overset{s_0}{b_0}\}$, $\overset{k}{b'} \in \{\overset{s_1}{b_1}\}$ we have

$$\begin{aligned}
& E^{\mathfrak{A}(k,q,n)}(I, \overset{k}{a})(I+1, \overset{k}{a'}) \\
\iff & (P^{\mathfrak{L}(k,q)} \overset{k}{a} \Leftrightarrow P^{\mathfrak{L}(k,q)} \overset{k}{a'}) \\
\iff & (P^{\mathfrak{R}(k,q)} \overset{k}{b} \Leftrightarrow P^{\mathfrak{R}(k,q)} \overset{k}{b'}) \qquad \text{(assumption case 2)} \\
\iff & (P^{\mathfrak{L}(k,q)} \overset{k}{b} \Leftrightarrow P^{\mathfrak{L}(k,q)} \overset{k}{b'}) \qquad \text{(definition of } \mathfrak{L}(k,q),\ \mathfrak{R}(k,q)\text{)} \\
\iff & E^{\mathfrak{B}(k,q,n)}(I, \overset{k}{b})(I+1, \overset{k}{b'})
\end{aligned}$$

□

The game

For notational convenience we let $n = 2^q + 2$, $m = m(k,q)$, $\mathfrak{A} = \mathfrak{A}(k,q,n)$, $\mathfrak{B} = \mathfrak{B}(k,q,n)$, $\mathfrak{L} = \mathfrak{L}(k,q)$, $\mathfrak{R} = \mathfrak{R}(k,q)$. (Remember that $q = qr(\chi^*) + k$.)
We have to prove:

4.4 Proposition : *Player $\exists$ wins the $(\chi^*, \mathfrak{B})$-game.*

Since $\mathfrak{A} \models \chi^*$ player $\exists$ has a winning strategy in the $(\chi^*, \mathfrak{A})$-game. Essentially she[1] is playing the $(\chi^*, \mathfrak{B})$-game by the same strategy. To describe this we are going to play both games parallel and obtain the moves of $\exists$ in the $(\chi^*, \mathfrak{B})$-game considering her moves in the $(\chi^*, \mathfrak{A})$-game. Note that this turns our game into a kind of Ehrenfeucht-Fraïssé game.

Remember that a situation in the $(\chi^*, \mathfrak{A})$-game (and the $(\chi^*, \mathfrak{B})$-game) consists of a current formula ψ and an interpretation of the free variables of ψ. As it was said before we are going to play both games parallel. The current formula will always be the same in both games. In each situation of the game we call those pairs of elements $a \in A$, $b \in B$ that are the interpretations of the same free variable of the current formula in the $(\chi^*, \mathfrak{A})$-game ($(\chi^*, \mathfrak{B})$-game respectively) *pairs of corresponding elements.* To simplify the notation we also consider pairs of elements that interpret the same constant in $\mathfrak{A}$ and $\mathfrak{B}$ as pairs of corresponding elements.
We will always speak of one situation to refer to the pair of situations we are in in the two games. It consists of the current formula and an interpretation of its free variables in both structures.

We are finished if we can prove that $\exists$ can preserve the following throughout the whole game:

(1) The pairs of corresponding elements form a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
(2) $\exists$ plays the $(\chi^*, \mathfrak{A})$-game according to her winning strategy.

Because then whenever the game arrives at an atomic or negated atomic formula it will hold in $\mathfrak{A}$ because of item (2) and thus in $\mathfrak{B}$ because of item (1). Thus $\exists$ wins the $(\chi^*, \mathfrak{B})$-game.

Informally $\exists$'s strategy is as follows:
She always copies the $\vee$, $\wedge$ and $(\neg)PFP$–moves from the $(\chi^*, \mathfrak{A})$-game. In rows near the first she just copies the moves of $\forall$ when she has to select new elements in $\forall$ or $\exists$–moves. In rows near the last she switches the components and then copies $\forall$'s moves. This is possible because $\mathfrak{R}$ and $\mathfrak{L}$ look very similar. There will always be an empty part in the middle so no troubles occur from mixing up the rows "near the first" and "near the last".
To realize this we are going to prove inductively that $\exists$ can not only preserve (1) and (2) but also the following in each situation of the game:

[1] We denote player $\exists$ by female and $\forall$ by male pronouns.

(3) If $a \in A,\ b \in B$ are corresponding elements, then both a and b are in the same row.

(4) Let r be the rank of the current situation in the $(\chi^*, \mathfrak{A})$-game and $(\chi^*, \mathfrak{B})$-game (which is always the same in both games because it only depends on the current formula). There exist numbers $M, N \in \{1, \ldots, n\}$ such that $M - N > 2^r$ and none of the corresponding elements is in a row I with $M < I < N$.

(5) For $1 \leq I \leq M$, if $(I, a_1), (I, b_1), \ldots, (I, a_s), (I, b_s)$ are the pairs of corresponding elements in row I, then the mapping defined by $a_j \mapsto b_j$ $(1 \leq j \leq s)$ is a partial isomorphism from $\mathfrak{L}$ to $\mathfrak{L}$, i.e. $\varphi^0_{\mathfrak{L},\overset{s}{a}} = \varphi^0_{\mathfrak{L},\overset{s}{b}}$.

(6) For $N \leq I \leq n$, if $(I, a_1), (I, b_1), \ldots, (I, a_s), (I, b_s)$ are the pairs of corresponding elements in row I, then the mapping defined by $a_j \mapsto b_j$ $(1 \leq j \leq s)$ is a partial isomorphism from $\mathfrak{L}$ to $\mathfrak{R}$, i.e. $\varphi^0_{\mathfrak{L},\overset{s}{a}} = \varphi^0_{\mathfrak{R},\overset{s}{b}}$.

Note that (1) follows from (2)–(6) because according to (4) there is always a gap between M and N. So on neighboured rows we have the same type of partial isomorphisms in (5) and (6) ($\mathfrak{L}$ to $\mathfrak{L}$ in rows $\leq M$ and $\mathfrak{L}$ to $\mathfrak{R}$ in rows $\geq N$). Thus Lemma 4.3 applies.

In the beginning (1)–(6) hold with $M = 1$ and $N = n$ because then the only pairs of corresponding elements are the constants in rows 1 and n.

Suppose we are in a situation of the games with current formula ψ and let M and N be chosen such that (1)–(6) hold. We have to define a formula ψ' to be the current formula in the next situation, an interpretation of the free variables of ψ' and M' and N' to be in accordance with (1)–(6). First we suppose we are not in an R–situation (This case will be treated later.)

$\psi = \psi_1 \vee \psi_2$: $\exists$ selects ψ_1 or ψ_2 to be the current formula in the next situation according to her winning strategy in the $(\chi^*, \mathfrak{A})$-game. The free variables are interpreted as they were before and M and N also remain unchanged.

$\psi = \psi_1 \wedge \psi_2$: $\forall$ selects ψ_1 or ψ_2 to be the current formula in the next situation. The free variables are interpreted as they were before and M and N remain unchanged.

$\psi = \exists x \psi'$: $\exists$ selects an $(I, a) \in A$ to be the interpretation of x in the next situation (with current formula ψ') according to his winning strategy in the $(\chi^*, \mathfrak{A})$-game. Hence (2) holds.

Case 1: $I - M \leq N - I$

Let $M' := max\{M, I\}$ and $N' := N$ and suppose $(I, a_1), (I, b_1), \ldots, (I, a_s), (I, b_s)$ are the pairs of corresponding elements in row I. Note that $s < q$ because there are at most k constants in a row and ψ' has got $\leq qr(\chi^*)$ free variables.

Since $\mathfrak{L} \models \Phi_q(\mathcal{P}_k)$ we have

$$\mathfrak{L} \models \forall \overset{s}{v}\ (\varphi^0_{\mathfrak{L},\overset{s}{a}} \rightarrow \exists v_{s+1}\ \varphi^0_{\mathfrak{L},\overset{s}{a}a})$$

Condition (5) implies $\varphi^0_{\mathfrak{L},\overset{s}{a}} = \varphi^0_{\mathfrak{L},\overset{s}{b}}$ so $\exists$ can select an element $(I, b) \in B$ to be the interpretation of the new variable x in the $(\chi^*, \mathfrak{B})$-game such that $\mathfrak{L} \models \varphi^0_{\mathfrak{L},\overset{s}{a}a}[\overset{s}{b}, b]$.

So (3) and (5) also hold in the next situation (at least for row I).

Case 2: $I - M > N - I$

Here (6) is the item we have to look for in particular:
Let $M' := M'$ and $N' := min\{I, N\}$ and
$(I, a_1), (I, b_1), \ldots, (I, a_s), (I, b_s)$ be the pairs of corresponding elements in row I.
Now $\exists$ can find a suitable element using

$$\mathfrak{R} \models \varphi^0_{\mathfrak{L},\overset{s}{a}} \rightarrow \exists v_{s+1}\, \varphi^0_{\mathfrak{L},\overset{s}{a}a}[\overset{s}{b}] \,.$$

Note that the rank of the new situation is 1 below the rank of the old situation; thus (4) holds by induction hypothesis and $N' - M' \geq \frac{N-M}{2}$. Since nothing changes in rows $\neq I$ items (3), (5), (6) hold in all rows.

$\psi = \forall\psi'$: $\forall$ selects $(I, b) \in B$ to be the interpretation of x in the $(\chi^*, \mathfrak{B})$-game. Similarily to the $\exists$–case we find an $(I, a) \in A$ to be the interpretation of x in the $(\chi^*, \mathfrak{A})$-game and M', N' such that (1)–(6) hold.

$\psi = (\neg)\ [PFP_{\overset{k}{x},X}\ \varphi]\ \overset{k}{u}$: $\exists$ selects an $i \geq 0$. The current formula in the next situation is $(\neg)\psi^*_i(\overset{k}{u})$. The current interpretation and M and N remain unchanged.

So finally we have to take care of the R–situations. Since the R–situations do not belong to the original game but were introduced later to indicate certain situations of the game we still have to perform one of the moves above. But before that we do the following:

Suppose we are in an R–situation R with current formula $\psi = (\neg)\exists\ \overset{l}{x}\ (\overset{l}{x} = \overset{l}{t} \wedge \psi_j)$. Let F be the corresponding F–situation with current formula ψ_F. The situation F must have occurred in the game before. Suppose M_F and N_F were suitable for (1)–(6) in that situation. Let r be the rank of the current situation and $r_F = r + l$ the rank of situation F. Essentially we want to step back to that situation.
Note that by Observation 2.5 the only variables that are free variables of ψ but not of ψ_F are the free variables of $\overset{l}{t}$. All other free variables of ψ are also free variables of ψ_F. Moreover their interpretation has not changed since the situation F and they are interpreted by elements in rows $\leq M_F$ or $\geq N_F$.

Let $a_1, b_1 \ldots, a_{l'}, b_{l'}$ be the current interpretations of the variables in $free(\overset{l}{t})$. They might be in rows between M_F and N_F. But by the pigeonhole–principle there exist M' and N' such that $M_F \leq M' < N' \leq N_F$,

$$N' - M' \geq \frac{N_F - M_F}{l' + 1} > \frac{2^{r_F}}{l+1} = \frac{2^{r+l}}{l+1} \geq 2^r$$

and none of the pairs $a_i, b_i (i \leq l')$ is in a row between N' and M'.

All variables are interpreted in the next situation as they were before thus M' and N' will be suitable in the next situation. Clearly (2), (3) and (4) hold.

But there seems to be a problem in proving (5) and (6):
Some of the $a_i, b_i (i \leq l')$ (without loss of generality we can assume all) might be in a row $\geq N'$, but $\leq M$ (case 1) or in a row $\leq M'$, but $\geq N$ (case 2). Thus they have "changed the components".
Consider for example the first case. By induction hypothesis (5) we only know that $\varphi^0_{\mathfrak{L}, \overset{l'}{a}} = \varphi^0_{\mathfrak{L}, \overset{l'}{b}}$. Now we need $\varphi^0_{\mathfrak{L}, \overset{l'}{a}} = \varphi^0_{\mathfrak{R}, \overset{l'}{b}}$ to prove (6) for the new situation.
But since $\chi^* \in PFP^{k-1}$ the arity of each fixed–point operator is $< k$ thus $l' \leq l < k$. Remembering that $\mathfrak{L}$ and $\mathfrak{R}$ only have a k–ary relation we immediately see that the atomic type of tuples of length $< k$ only depends on their equality type. Thus $\varphi^0_{\mathfrak{L}, \overset{l'}{b}} = \varphi^0_{\mathfrak{R}, \overset{l'}{b}}$ holds.
Note that this is one of the crucial points of the whole proof! Here we use the "homogeneity of the structures $\mathfrak{L}$ and $\mathfrak{R}$ in the $(k-1)$–tuples".

Once we have done this we turn to the "regular move" of the R–situation. Remembering that the current formula is $(\neg)\exists \overset{l}{x} (\overset{l}{x} = \overset{l}{t} \wedge \psi_j)$ we continue with an $\exists$ or $\forall$–move. $\square$

As we have already claimed at the end of Section 2 we get:

4.5 Corollary : *The hierarchies* $(\mathbf{TC}^k)_{k\geq 1}$, $(\mathbf{EFP}^k)_{k\geq 1}$, $(\mathbf{LFP}^k)_{k\geq 1}$, $(\mathbf{IFP}^k)_{k\geq 1}$ *are strict.*

Open problem: *Our proof essentially needs a new signature for each k. It would be a desireable extension to prove the hierarchy theorem for a uniform signature.*

Added in proof: Meanwhile I solved this problem. The proof will appear elsewhere.

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