Complete Problems for Fixed–Point Logics

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1 Introduction

The notion of logical reducibilities is derived from the idea of interpretations between theories. It was used by Lovász and Gács [LG77] and Immerman [Imm87] to give complete problems for certain complexity classes and hence establish new connections between logical definability and computational complexity.
However, the notion is also interesting in a purely logical context. For example, it is helpful to establish non-expressibility results.

We say that a class $\mathcal{C}$ of τ–structures is a *complete problem* for a logic $\tilde{\mathbf{L}}$ under $\mathbf{L}$–reductions if it is definable in $\tilde{\mathbf{L}}[\tau]$ and if every class definable in $\tilde{\mathbf{L}}$ can be "translated" into $\mathcal{C}$ by L–formulae (cf. Section 4).

We prove the following theorem:

1.1 Theorem : *There are complete problems $\mathcal{P}$ for partial fixed–point logic and $\mathcal{I}$ for inductive fixed–point logic under quantifier–free reductions.*

The main step of the proof is to establish a new normal form for fixed–point formulae (which might be of some interest itself). To obtain this normal form we use theorems of Abiteboul and Vianu [AV91a] that show the equivalence between the fixed–point logics we consider and certain extensions of the database query language Datalog.

In [Dah87] Dahlhaus gave a complete problem for least fixed–point logic. Since least fixed–point logic equals inductive fixed–point logic by a well–known result of Gurevich and Shelah [GS86] this already proves one part of our theorem.

However, our class $\mathcal{I}$ gives a natural description of the fixed–point process of an *inductive* fixed–point formula and hence sheds some light on completely different aspects of the logic than Dahlhaus's construction, which is strongly based on the features of *least* fixed–point formulae.

For the restriction to ordered structures analogous results are known from descriptive complexity theory (cf. [Imm87, MP93]).

Logical reductions are tightly connected with the concept of Lindström–quantifiers. We refer the reader to [Daw93] for detailed discussion of this connection.
We only want to mention here that if there exists a complete problem for $\tilde{\mathbf{L}}$ under $\mathbf{L}$–reductions then $\tilde{\mathbf{L}}$ can be captured by $\mathbf{L}$ augmented by a uniform sequence of Lindström–quantifiers.
Thus our theorem implies that partial and inductive fixed–point logic can be captured by first–order logic augmented by uniform sequences of Lindström–quantifiers.

Recently Hella [Hel94] proved a stronger version of Dahlhaus's result (and hence of one part of our theorem); he gave a Lindström–quantifier (in fact a complete problem) for least fixed–point logic augmented by monotone Lindström–quantifiers. As Dahlhaus's, Hella's construction makes essential use of the monotonicity of least fixed–point processes.

We only consider finite structures. Our results do not hold in the scope of all structures; in fact not even the results in [AV91a] and [GS86] on which our proofs are based do.
For technical reasons we assume that each structure contains at least two elements.

Our notation is standard based on [EFT94]. The only thing that should be mentioned is that we denote tuples in the form $x_1 \ldots x_k$ and abbreviate them by $\overset{k}{x}$ or $\bar{x}$. For example, this allows us to write $\overset{k}{x}\overset{l}{y}$ for the tuple $x_1 \ldots x_k y_1 \ldots y_l$.

2 Fixed–point logics

We are going to work with a generalization of the fixed–point logics that allows simultaneous inductions. It is known that this does not increase the expressive power but it helps us in defining the normal form.

Partial fixed–point logic

2.1 Definition : (1) The class *S–PFP* of *simultaneous partial fixed–point formulae* is given by means of the calculus consisting of the first–order rules and the rule

$$(S\text{–}PFP) \quad \frac{\varphi_1, \ldots, \varphi_m}{[S\text{–}PFP_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \varphi_1, \ldots, \varphi_m]\bar{u}}$$

where $m \geq 1$ and, for each $i \leq m$, X_i is a relation variable whose arity matches the length of $\bar{x}_i$, and $\bar{u}$ is a tuple of constants of the same length as $\bar{x}_1$.

(2) The class *PFP* of partial fixed–point formulae is the subclass of *S–PFP* allowing no simultaneous inductions, i.e. allowing the S–PFP–rule only for $m = 1$. □

To define the semantics for each simultaneous partial fixed–point formula

$$[S\text{–}PFP_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \varphi_1, \ldots, \varphi_m]\bar{u}$$

and interpretation $\mathfrak{I} = (\mathfrak{A}, \alpha)$ we define sequences $(X_{ji}^{\mathfrak{I}})_{i \geq 0}$ $(1 \leq j \leq m)$ of relations on A by

$$X_{j0}^{\mathfrak{I}} := \emptyset$$

$$X_{j(i+1)}^{\mathfrak{I}} := \{\bar{a} \in A \mid \mathfrak{I} \models \varphi_j[\bar{a}, X_{1i}^{\mathfrak{I}}, \ldots, X_{mi}^{\mathfrak{I}}]\}$$

$$X_{j\infty}^{\mathfrak{I}} := \begin{cases} X_{jk}^{\mathfrak{I}} & \text{where } k = min\{i \mid \bigwedge_{j'=1}^{m} X_{j'i}^{\mathfrak{I}} = X_{j'(i+1)}^{\mathfrak{I}}\} \\ & \text{(if such a } k \text{ exists)} \\ \emptyset & \text{otherwise} \end{cases}$$

We let

$$\mathfrak{I} \models [S\text{–}PFP_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \varphi_1, \ldots, \varphi_m]\bar{u} \iff \alpha(\bar{u}) \in X_{1\infty}^{\mathfrak{I}}$$

and define the semantics of the logic **S–PFP** inductively.
Hence our fixed–point formula defines the relation $X_{1\infty}^{\mathfrak{I}}$; we consider X_1 as our *goal–predicate* (thinking of Datalog–programs).

Inductive and least fixed–point logic

Replacing *PFP* by *IFP* in *(S)–PFP*–formulae we obtain the class *(S)–IFP* of *(simultaneous) inductive fixed–point formulae.*
So the syntax of *IFP* and *PFP*–formulae is essentially the same, we just give different names to the operators.

But the semantics is different:
Let $\mathfrak{I} = (\mathfrak{A}, \alpha)$ be an interpretation and consider the *S–IFP*–formula

$$[S\text{–}IFP_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \varphi_1, \ldots, \varphi_m]\bar{u} \ .$$

Here we define sequences $(X_{ji}^{\mathfrak{I}})_{i \geq 0}$ $(1 \leq j \leq m)$ of relations on A by

$$\begin{aligned}
X_{j0}^{\mathfrak{I}} &:= \emptyset \\
X_{j(i+1)}^{\mathfrak{I}} &:= X_{ji}^{\mathfrak{I}} \cup \{\bar{a} \in A \mid \mathfrak{I} \models \varphi_j[\bar{a}, X_{1i}^{\mathfrak{I}}, \ldots, X_{mi}^{\mathfrak{I}}]\} \\
X_{j\infty}^{\mathfrak{I}} &:= \bigcup_{i \geq 0} X_{ji}^{\mathfrak{I}} = X_{jk}^{\mathfrak{I}} \quad \text{where } k = min\{i \mid \textstyle\bigwedge_{j'=1}^{m} X_{j'i}^{\mathfrak{I}} = X_{j'(i+1)}^{\mathfrak{I}}\}
\end{aligned}$$

Again we let our fixed–point formula define $X_{1\infty}^{\mathfrak{I}}$.

Finally, the class *LFP* of *least fixed–point formulae* is the subclass of *IFP* where the fixed–point operator $[IFP_{\bar{x}, X} \ldots]$ is only applied to formulae where X only occurs positively.
Gurevich and Shelah [GS86] proved that this does not reduce the expressive power, i.e. **LFP** = **IFP**.

3 A normal form for fixed–point formulae

Most of the work has already been done by Abiteboul and Vianu [AV91a]. They proved that every *PFP*–formula is equivalent to a program in an extension of the query language datalog. The following stronger version of their theorem can be found in [EF].
In our notation it looks as follows:

3.1 Theorem ([AV91a, EF]) : *Every PFP–formula (with at least one free variable) is equivalent to an S–PFP–formula* χ *of the form*

$$[S\text{–}PFP_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \alpha_1, \ldots, \alpha_m]\bar{u}$$

where each α_i $(i \leq m)$ is a disjunction of formulae of the form

$$\exists \bar{y}(\theta_1 \wedge \ldots \wedge \theta_n)$$

with atomic or negated atomic θ_i $(i \leq n)$.
Furthermore, none of the free variables of χ occur in an α_i $(i \leq m)$.

The following definition gives our normal form for *PFP*–formulae. The first part is needed for technical reasons.

3.2 Definition : (1) A formula $\chi = [PFP_{\overset{k}{x},X}\varphi]\overset{k}{u}$ is called *modest*, if for each interpretation $\mathfrak{I} = (\mathfrak{A}, \alpha)$ and for all $i \geq 1$ we have $\emptyset \subsetneq X_i^{\mathfrak{I}} \subsetneq A^k$.
If χ is modest, for convenience we also call the formula $\forall v \chi$ modest.

(2) We say that a *PFP*–formula χ is in *normal form* if it is of the form

$$\forall v \big[PFP_{\bar{x},X}\ \psi_0 \vee \exists \bar{y} \in X\ \exists \bar{z} \notin X\ \psi\big]\bar{u}$$

where:

- The formulae ψ_0 and ψ are quantifier–free.
- Neither X, nor v, nor any of the free variables of the whole formula occur in ψ_0 or ψ.
- The formula is modest.

□

3.3 Lemma : *Every PFP–formula is equivalent to a formula in normal form*

Proof: To be able to apply Theorem 3.1 correctly, during the proof we only consider *PFP*–formulae that have at least one free variable. The last step of the proof shows that this is no real restriction.

In the **first step** of the proof we show that every *PFP*–formula is equivalent to a formula of the form $\forall v[PFP_{\bar{x},X}\varphi]\bar{u}$ where φ is existential first–order and neither v nor any of the free variables of the formula occur in φ.

To see this we consider a formula

$$\chi = [S\text{–}PFP_{\bar{x}_1,X_1,\ldots,\bar{x}_m,X_m}\alpha_1, \ldots, \alpha_m]\bar{u}$$

of the form obtained by Theorem 3.1.

To replace the simultaneous induction by a single one we use well–known arguments which are due to Moschovakis [Mos74].

Let $v_1, \dots, v_m$ be variables that do not occur in χ. Without loss of generality we can assume that all X_i ($i \le m$) have the same arity r (hence the length of the $\bar{x}_i$ and $\bar{u}$ is also r). Let X be an $(r+1)$–ary relation symbol.
Consider

$$\varphi := \big(x_{r+1} = v_1 \wedge \beta_1\big) \vee \dots \vee \big(x_{r+1} = v_m \wedge \beta_m\big)$$

where β_i is the result of replacing each subformula of the form $X_j \overset{r}{t}$ in α_i by $X \overset{r}{t}\, v_j$.
Then an easy induction shows that

$$\chi \dashv\models \exists v_1 \dots \exists v_m \big(\bigwedge_{i<j\le m} v_i \ne v_j \wedge [PFP_{\overset{r+1}{x},X}\varphi]\bar{u}v_1\big)$$

in structures of cardinality $\ge m$.

Next, we want to remove the variables $v_1, \dots, v_m$. Note that they are only used to distinguish between the different processes in the original simultaneous induction. But this can also be done using equality types of tuples of two elements (which is a standard trick); this only increases the arity of the relation variable X. We encode v_1 by the diagonal, i.e. those tuples of elements which are equal in each component.
To illustrate this let us, for example, assume that $m = 3$. We encode v_1 by the type aaa, v_2 by the type aab, and v_3 by the type aba. Instead of $(r+1)$–ary we now let X be $(r+3)$–ary and replace (in $\alpha_1, \dots, \alpha_3$) each subformula of the form $X_1 \overset{r}{t}$ by $\exists x X \overset{r}{t}\, xxx$, each subformula of the form $X_2 \overset{r}{t}$ by $\exists x, y(x \ne y \wedge X \overset{r}{t}\, xxy)$, and each subformula of the form $X_3 \overset{r}{t}$ by $\exists x, y(x \ne y \wedge X \overset{r}{t}\, xyx)$. Thereby we obtain formulae $\beta_1', \dots, \beta_3'$.
We let

$$\begin{aligned} \varphi' := \quad & (x_{r+1} = x_{r+2} \wedge x_{r+1} = x_{r+3} \wedge \beta_1') \\ & \wedge(x_{r+1} = x_{r+2} \wedge x_{r+1} \ne x_{r+3} \wedge \beta_2') \\ & \wedge(x_{r+1} \ne x_{r+2} \wedge x_{r+1} = x_{r+3} \wedge \beta_3') \end{aligned}$$

We proceed completely analogous for arbitrary m. Say, s is the appropriate arity for the relation variable X. Then we obtain an existential first–order formula φ' such that

$$\chi \dashv\models \forall v[PFP_{\overset{s}{x},X}\varphi']\bar{u}\, v \dots v =: \chi' ,$$

even in structures of cardinality ≥ 2. Hence we are finished with the first step.

We call a *PFP*–formula *(k,l)–bounded* if it is of the form

$$\forall v[PFP_{\bar{x},X}\ \psi_0\ \vee \exists \bar{y}_1 \in X \dots \exists \bar{y}_k \in X\ \ \exists \bar{z}_1 \notin X \dots \exists \bar{z}_l \notin X\ \ \psi]\bar{u}$$

where ψ_0 is quantifier–free and ψ is existential first–order, neither X, nor v, nor any of the free variables of the formula occur in ψ_0 or ψ, and the formula is modest.

In the **second step** we show that χ' is equivalent to formula ξ which is (k,k)–bounded for some $k \geq 1$.

We first replace, one after another, each positive occurrence of a subformula $(X\ \overset{s}{t})$ in φ' by $(\exists\ \overset{s}{z}\in X\ \ \overset{s}{z}=\overset{s}{t})$ and each occurrence of a subformula $(\neg X\ \overset{s}{t})$ in φ' by $(\exists\ \overset{s}{z}\notin X\ \ \overset{s}{z}=\overset{s}{t})$. Here we use new variables $\overset{s}{z}$ in each step.

Now the rest seems to be easy. Since there are no universal quantifiers or negation–symbols in front of our relativized quantifiers $\exists\ \overset{s}{z}\in X$ and $\exists\ \overset{s}{z}\notin X$ and all these quantifiers speak of different variables we seem to be able to move them to the front of φ'. For example on the first sight the equivalence

$$\xi \vee \exists\ \overset{s}{z}\in X\ \xi' \dashv\models \exists\ \overset{s}{z}\in X\ (\xi \vee \xi')$$

is correct as long as no z_i occurs in ξ. Unfortunately it is not because X may be empty. (The same kind of trouble may occur in the negated case if X contains all tuples.)

To circumvent this problem we are going to define a formula χ'' that is equivalent to χ' and modest.
Therefore we increase the arity of X by 2 (meanwhile it is $(s+2)$–ary), replace each subformula $X\ \overset{s}{t}$ in φ' by $X\ \overset{s}{t}\ t_1 t_1$, and obtain a formula φ''. We let $\psi_0 = x_{s+1} \neq x_{s+2} \wedge x_s = x_{s+2}$. Consider the formula

$$\chi'' = \forall v[PFP_{\overset{s+2}{x},X}\psi_0 \vee (x_{s+1} = x_{s+2} \wedge \varphi'')]\bar{u}\, v \dots v\, vv$$

It is clearly equivalent to χ' because its truth value only depends on those tuples in the fixed point whose last two places are equal, and by the construction of χ'' these are exactly the tuples whose first s places form a tuple in the fixed point of χ'.
It is also not hard to see that χ'' is modest since (in each interpretation) $\psi_0 \vee (x_{s+1} = x_{s+2} \wedge \varphi'')$ holds for all tuples (interpreting $\overset{s+2}{x}$) whose last three places

are of equality type aba, but it does not hold for any tuple whose last three places are of equality type bba.

So during the fixed–point process of χ'' the relation X is never empty and it never contains all tuples. Hence we can move the relativized quantifiers to the front of $(x_{s+1} = x_{s+2} \wedge \varphi'')$ and obtain a formula

$$\xi = \forall v[PFP_{\bar{x},X}\psi_0 \vee \exists \bar{y}_1 \in X \ldots \exists \bar{y}_k \in X \exists \bar{z}_1 \notin X \ldots \exists \bar{z}_l \notin X \ \ \psi]\bar{u}v\ldots v$$

that is (k, l)–bounded and equivalent to χ.

Note that ξ is still modest; in particular this implies that we can increase k and l artificially. So we can assume without loss of generality that $k = l \geq 1$ and the second step is done.

In the **third step** we prove that ξ is equivalent to a $(1, 1)$—bounded formula.

Let us (re)define r to be the arity of X and let Y be a new $(2kr+1)$–ary relation variable. Let $w_1, \ldots, w_{2k}$ be individual variables that do not occur in ξ.
We let

$$\begin{aligned}\xi' = \exists w_1, \ldots, w_{2k}\Big(& \textstyle\bigwedge_{i<j\leq 2k} w_i \neq w_j \\ & \wedge \forall v\Big[FP_{x_0\bar{x}_1\ldots\bar{x}_{2k},Y} \quad (x_0 = w_1 \wedge \psi_0(\bar{x}_1)) \\ & \qquad \vee \exists y_0\bar{y}_1\ldots\bar{y}_{2k} \in Y \exists z_0\bar{z}_1\ldots\bar{z}_{2k} \notin Y \\ & \qquad\qquad \textstyle\bigvee_{i=1}^{2k}(x_0 = w_i \wedge \psi_i)\Big] w_1\bar{u}v\ldots v\Big)\end{aligned}$$

where

$$\psi_1 = \Big(y_0 = w_{2k} \wedge \psi(\bar{x}_1, \bar{y}_1, \ldots, \bar{y}_{2k})\Big),$$

$$\psi_{2i} = \Big(y_0 = w_{2i-1} \wedge z_0 = y_0 \wedge \bigwedge_{j=2}^{2k} \bar{z}_j = \bar{y}_j \wedge \bar{x}_{k+i} = \bar{z}_1 \wedge \bigwedge_{\substack{1\leq j\leq 2k,\\ j\neq k+i}} \bar{x}_j = \bar{y}_j\Big)$$

for $1 \leq i \leq k$, and

$$\psi_{2i+1} = \Big(y_0 = w_{2i} \wedge z_0 = y_0 \wedge \bigwedge_{\substack{1\leq j\leq 2k,\\ j\neq k+1}} \bar{z}_j = \bar{y}_j \wedge \bar{x}_i = \bar{z}_{k+1} \wedge \bigwedge_{\substack{1\leq j\leq 2k,\\ j\neq i}} \bar{x}_j = \bar{y}_j\Big)$$

and for $1 \leq i \leq k-1$.

To prove that ξ' is equivalent to ξ we let $\mathfrak{I} = (\mathfrak{A}, \alpha)$ be an interpretation for ξ' such that for $c_i := \alpha(w_i)$ we have: $\bigwedge_{i<j\leq 2k} c_i \neq c_j$.

Let $X_m := X_m^{\mathfrak{I}}$ and $Y_m := Y_m^{\mathfrak{I}}$ (for $m \geq 0$) be the stages of the fixed–point processes of ξ and ξ' respectively and recall that $\emptyset \subsetneq X_m \subsetneq A^r$ for each $m \geq 1$ (since ξ is modest).

For brevity we let, for $m \geq 0, i \leq 2k$,

$$Y_m c_i \_\_ := \{\bar{a}_1 \ldots \bar{a}_{2k} \in A \mid Y_m c_i \bar{a}_1 \ldots \bar{a}_{2k}\}$$

Claim: For all $m \geq 0$ and $1 \leq i \leq k$ we have:

$$(2i-1) \quad Y_{2km+2i-1} c_{2i-1} \_\_ = \underbrace{X_{m+1} \times \ldots \times X_{m+1} \times X_{m+1}}_{i \text{ times}} \times \underbrace{A^r \times \ldots \times A^r}_{(k-i) \text{ times}} \times \underbrace{X_{m+1}^C \times \ldots \times X_{m+1}^C}_{(i-1) \text{ times}} \times \underbrace{A^r \times A^r \times \ldots \times A^r}_{(k-i+1) \text{ times}}$$

$$(2i) \quad Y_{2km+2i} c_{2i} \_\_ = \underbrace{X_{m+1} \times \ldots \times X_{m+1}}_{i \text{ times}} \times \underbrace{A^r \times \ldots \times A^r}_{(k-i) \text{ times}} \times \underbrace{X_{m+1}^C \times \ldots \times X_{m+1}^C}_{i \text{ times}} \times \underbrace{A^r \times \ldots \times A^r}_{(k-i) \text{ times}}$$

and for all $i \leq 2k, j \leq 2k-1$

$$(*) \quad Y_{2km+i} c_i \_\_ = Y_{2km+i+j} c_i \_\_ \,.$$

The proof is by induction over m:

***m* =0:** (1)–($2k$) are proved inductively:

Since $Y_1 = \{a_0 \bar{a}_1 \ldots \bar{a}_{2k} \in A \mid a_0 = c_1, \overbrace{\mathfrak{I} \models \psi_0[\bar{a}_1]}^{\Longleftrightarrow X_1 \bar{a}_1}\}$, (1) holds for $m = 0$.

Suppose now that $(2i-1)$ is proved and observe that if $Y_{2i-1} c_{2i-1} \bar{a}_1 \ldots \bar{a}_{2k}$, $\neg Y_{2i-1} c_{2i-1} \bar{b}_1 \ldots \bar{b}_{2k}$, and $\bar{a}_j = \bar{b}_j$ for all $j \geq 2$ then $\neg X_1 \bar{b}_1$ (because otherwise we would also have $Y_{2i-1} c_{2i-1} \bar{b}_1 \ldots \bar{b}_{2k}$, a contradiction).

On the other hand, if $\neg X_1 \bar{b}$ then by induction hypothesis $(2i-1)$ there exist tuples $\bar{a}_1 \ldots \bar{a}_{2k}, \bar{b}_1 \ldots \bar{b}_{2k} \in A$ as above such that $\bar{b} = \bar{b}_1$.

This shows that $Y_{2i} c_{2i} \bar{a}_1 \ldots \bar{a}_{2k}$ if and only if $\neg X_1 \bar{a}_{k+i}$ and, by induction hypothesis $(2i-1)$, $\bigwedge_{j=1}^{i} X_1 \bar{a}_j$ and $\bigwedge_{j=1}^{i-1} \neg X_1 \bar{a}_{k+j}$.

We proceed similarly to prove $(2i+1)$:

$X_1 \bar{b}$ holds if and only if there exist tuples $\bar{a}_1 \ldots \bar{a}_{2k}, \bar{b}_1 \ldots \bar{b}_{2k} \in A$ such that $Y_{2i} c_{2i} \bar{a}_1 \ldots \bar{a}_{2k}$, $\neg Y_{2i} c_{2i} \bar{b}_1 \ldots \bar{b}_{2k}$, $\bar{a}_j = \bar{b}_j$ for all $j \neq k+1$, and $\bar{b} = \bar{b}_{k+1}$.

(∗) can easily be seen considering that $Y_i c_{2k}\_\_$ is empty until $i = 2k$, so $Y_i c_1\_\_$ does not change before the $(2k+1)$th step. Thus $Y_i c_2\_\_$ does not change (after its first change in the second step) before the $(2k+2)$th step, and so on.

$\boldsymbol{m \to m{+}1}$**:** To prove (1) for $m+1$ instead of m recall that $\bar{a} \in X_{m+2}$ if and only if

$$\exists \bar{a}_1, \ldots, \bar{a}_k \in X_{m+1}, \bar{b}_1, \ldots, \bar{b}_k \notin X_{m+1} : \mathfrak{I} \models \psi[\bar{a}, \bar{a}_1, \ldots, \bar{a}_k, \bar{b}_1, \ldots, \bar{b}_k] \ .$$

By induction hypothesis we have

$$Y_{2km} c_{2k}\_\_ = \underbrace{X_{m+1} \times \ldots \times X_{m+1}}_{k\ times} \times \underbrace{X^C_{m+1} \times \ldots \times X^C_{m+1}}_{k\ times}$$

Thus (1) is holds for $m+1$ by the definition of ψ_1. (2)–($2k$) follow by an induction completely analogous to the case $m = 0$.

(∗) can also be seen analogously to that case.

So the claim is proved, and as a consequence we have:

$$\mathfrak{I} \models \xi \Longleftrightarrow \mathfrak{I} \models \xi' \ .$$

Again we can use equality types of tuples build of two variables instead of the "indicator"–variables $w_1, \ldots, w_{2k}$ (the same trick was used in the first step) and thus obtain a formula

$$\xi'' := \forall v[PFP_{\bar{x},Z}\ \psi_0(\bar{x}) \vee \exists \bar{y} \in Z \exists \bar{z} \notin Z \psi(\bar{x}, \bar{y}, \bar{z})] v \ldots v\bar{u}v \ldots v$$

that is (1,1)–bounded and equivalent to ξ.

In the **fourth step** we take care of the existential quantifiers in ψ.

Let r be the arity of Z and suppose $\psi = \exists v_1 \ldots \exists v_s \psi'(\overset{r}{x}, \overset{r}{y}, \overset{r}{z}, v_1, \ldots, v_s)$ for a quantifier free ψ'. Let X be an $(r+s)$–ary relation variable and ξ''' the formula

$$\forall v[PFP_{\overset{r+s}{x},X}\ \psi_0(\overset{r}{x}) \vee \exists \overset{r+s}{y} \in X \exists \overset{r+s}{z} \notin X \psi'(\overset{r}{x}, \overset{r}{y}, \overset{r}{z}, y_{r+1}, \ldots, y_{r+s})] v \ldots v\bar{u}v \ldots v \ .$$

An easy induction shows that for each interpretation $\mathfrak{I} = (\mathfrak{A}, \alpha)$ and for each $i \geq 0$ we have:

$$X_i^{\mathfrak{I}} = Z_i^{\mathfrak{I}} \times A^s \ .$$

Thus $\xi'' \dashv\models \xi'''$ and the fourth step is done.

Recall that up to now the lemma is only proved for formulae with at least one free variable. In the **last step** we take care of sentences.

For *PFP*–sentences χ we consider the formula $\chi' = \chi \wedge w = w$ (where w is a variable that does not occur in χ) and note that $\chi \dashv\models \exists w \chi' \dashv\models \forall w \chi'$.
Our construction yields a formula of the form $\forall v[PFP \ldots]\bar{u}$ (where $\bar{u}$ is a tuple consisting of v, w, and maybe constants, and where neither v nor w occur in the scope of the *PFP*–operator) that is equivalent to χ'. Clearly, if we replace w by v in this formula we obtain a sentence of the desired form that is equivalent to χ. ■

3.4 Theorem [AV91a] : *Every IFP–formula (with at least one free variable) is equivalent to an S–IFP–formula of the form*

$$[S\text{–}IFP_{\bar{x}_1, X_1, \ldots, \bar{x}_m, X_m} \alpha_1, \ldots, \alpha_m]\bar{u}$$

where each α_i $(i \leq m)$ *is a disjunction of formulae of the form*

$$\exists \bar{y}(\theta_1 \wedge \ldots \wedge \theta_n)$$

with atomic or negated atomic θ_i $(i \leq n)$.
Furthermore, none of the free variables of the formula occur in any α_i $(i \leq m)$.

Replacing *PFP* by *IFP* in Definition 3.2 we define the normal form for *IFP*–fomulae. Completely analogous to Lemma 3.3 we obtain:

3.5 Lemma : *Every IFP–formula is equivalent to an IFP–formula in normal form.*

4 Complete Problems

We need some new notation here:

- By $\overset{r\times k}{x}$ we denote the r–tuple of k–tuples

$$\overset{k}{x}_1 \ldots \overset{k}{x}_r = x_{11} \ldots x_{1k} \; \ldots \; x_{r1} \ldots x_{rk}$$

- If $\varphi(\overset{k}{x})$ is a formula with free variables among $x_1, \ldots, x_k$ and $\mathfrak{A}$ a structure we let
$$\varphi(\overset{k}{x})^{\mathfrak{A}} = \{\overset{k}{a} \in A^k \mid \mathfrak{A} \models \varphi[\overset{k}{a}]\} .$$

4.1 Definition : (1) Let $\mathbf{L} = (L, \models)$ be a logic and σ and $\tau = \{R_1, \ldots, R_s\}$ two signatures (where R_i is an r_i-ary relation symbol (for $i = 1, \ldots, s$)).
An $\mathbf{L}$*–interpretation* of τ in σ is a sequence of $L[\sigma]$–formulae

$$\big(\varphi_1(\overset{r_1 \times k}{x}), \ldots, \varphi_s(\overset{r_s \times k}{x})\big)$$

(where the free variables of φ_h are in $\{x_{ij} \mid i \le r_h, j \le k\}$) for some $k \ge 1$.

(2) An $\mathbf{L}$–interpretation $\Phi = \big(\varphi_1(\overset{r_1 \times k}{x}), \ldots, \varphi_s(\overset{r_s \times k}{x})\big)$ associates with each σ–structure $\mathfrak{A}$ a τ–structure

$$\mathfrak{A}^{\Phi} = \Big(A^k, \varphi_1(\overset{r_1 \times k}{x})^{\mathfrak{A}}, \ldots, \varphi_s(\overset{r_s \times k}{x})^{\mathfrak{A}}\Big) .$$

We say that Φ is an $\mathbf{L}$–reduction of a class $\mathcal{C}$ of σ–structures to a class $\mathcal{D}$ of τ–structures if for all σ–structures $\mathfrak{A}$ we have

$$\mathfrak{A} \in \mathcal{C} \quad \Longleftrightarrow \quad \mathfrak{A}^{\Phi} \in \mathcal{D} .$$

(3) We say that a class $\mathcal{C}$ of τ–structures is *hard for* a logic $\tilde{\mathbf{L}}$ *under* $\mathbf{L}$*–reductions* if for every signature σ and every sentence $\varphi \in \tilde{L}[\sigma]$ there exists an $\mathbf{L}$–reduction of $Mod(\varphi)$ to $\mathcal{C}$.
$\mathcal{C}$ is *complete for* $\tilde{\mathbf{L}}$ *under* $\mathbf{L}$*–reductions* if it is definable in $\tilde{\mathbf{L}}$ and hard for $\tilde{\mathbf{L}}$ under $\mathbf{L}$–reductions.
Occasionally we say that $\mathcal{C}$ is a *complete problem* for $\tilde{\mathbf{L}}$ (under $\mathbf{L}$–reductions).

(4) In particular, if $\mathbf{L}$ is the logic whose formulae are the quantifier-free formulae we say that $\mathcal{C}$ is complete (hard, a complete problem) for $\tilde{\mathbf{L}}$ under *quantifier–free reductions.* □

The classes $\mathcal{P}$ and $\mathcal{I}$

For the rest of this paper we let $\tau = \{T, L, R\}$ with ternary T and unary L, R. Each τ–structure $\mathfrak{A}$ gives rise to a sequence $(P_i^{\mathfrak{A}})_{i \ge 1}$ defined by $P_1^{\mathfrak{A}} = L^{\mathfrak{A}}$ and

$$P_{i+1}^{\mathfrak{A}} = \{a \in A \mid \exists b \in P_i^{\mathfrak{A}}, c \notin P_i^{\mathfrak{A}} : T^{\mathfrak{A}} abc\}$$

for each $i \geq 1$. We let

$$P_\infty^{\mathfrak{A}} = \begin{cases} P_m^{\mathfrak{A}} & \text{where } m = \min\{i \mid P_i^{\mathfrak{A}} = P_{i+1}^{\mathfrak{A}}\} \text{ if it exists} \\ \emptyset & \text{otherwise} \end{cases}$$

Note that if there exists an $i \geq 1$ such that $P_i^{\mathfrak{A}} = \emptyset$ or $P_i^{\mathfrak{A}} = A$ then $P_{i+1}^{\mathfrak{A}} = \emptyset$ thus $P_\infty^{\mathfrak{A}} = \emptyset$.

Let $\mathcal{P}$ be the class consisting of all τ–structures $\mathfrak{A}$ with $R^{\mathfrak{A}} \subseteq P_\infty^{\mathfrak{A}}$.
The class $\mathcal{P}$ mirrors what happens in the fixed–point process of a *PFP*–sentence in normal form.

Let

$$\chi := \forall v[PFP_{\overset{k}{x},X}\ \psi_0(\overset{k}{x}) \vee \exists\, \overset{k}{y} \in X\ \exists\, \overset{k}{z} \notin X\ \psi(\overset{k}{x}, \overset{k}{y}, \overset{k}{z})]\ \overset{k}{u}$$

be such a sentence and let $\theta(\overset{k}{x})$ be the quantifier–free type of $\overset{k}{u}$. Then, since v does not occur in the scope of the *PFP*–operator, we have

$$\chi \mathrel{=\!\mid\!\models} \forall\, \overset{k}{x}\ (\theta(\overset{k}{x}) \longrightarrow [PFP_{\overset{k}{x},X}\ \psi_0(\overset{k}{x}) \vee \exists\, \overset{k}{y} \in X\ \exists\, \overset{k}{z} \notin X\ \psi(\overset{k}{x}, \overset{k}{y}, \overset{k}{z})]\ \overset{k}{x})\ .$$

Consider the quantifier–free interpretation

$$\Phi := \Big(\psi(\overset{k}{x}, \overset{k}{y}, \overset{k}{z}) \vee \psi_0(\overset{k}{x}), \psi_0(\overset{k}{x}), \theta(\overset{k}{x})\Big)$$

of τ in σ (where σ is the signature of χ).

An easy induction (using that χ is modest) shows that for each $i \geq 1$ we have $P_i^{\mathfrak{A}^\Phi} = X_i^{\mathfrak{A}}$. Thus

$$\mathfrak{A} \models \chi \quad \Longleftrightarrow \quad \mathfrak{A} \in \mathcal{P}\ .$$

Since by Lemma 3.3 every *PFP*–sentence is equivalent to a sentence in normal form we have just proved that $\mathcal{P}$ is hard for **PFP** under quantifier–free reductions.

Furthermore, $\mathcal{P}$ is the class of models of the fixed–point sentence

$$\xi = \forall u\big(Ru \longrightarrow [PFP_{x,X}\ (Lx \wedge X = \emptyset) \vee \exists y \in X, \exists z \notin XTxyz]u\big)$$

hence complete for **PFP**.
To see this $\mathfrak{A}$ be a τ–structure and consider the sequences $(P_i^{\mathfrak{A}})_{i \geq 1}$ and $(X_i^{\mathfrak{A}})_{i \geq 0}$. If there exists no $i \geq 1$ such that $P_i^{\mathfrak{A}} = \emptyset$ or $P_i^{\mathfrak{A}} = A$ then for each $i \geq 1$ we have $P_i^{\mathfrak{A}} = X_i^{\mathfrak{A}}$. Otherwise let i_0 be the first such i. We still have $P_i^{\mathfrak{A}} = X_i^{\mathfrak{A}}$

for each $i \leq i_0$. We have seen above that $P_\infty^{\mathfrak{A}} = \emptyset$ in this case; and we have $X_{i_0+1}^{\mathfrak{A}} = \emptyset$ (if $X_{i_0}^{\mathfrak{A}} = P_{i_0}^{\mathfrak{A}} = A$) or $X_{i_0+1}^{\mathfrak{A}} = L^{\mathfrak{A}}$ (if $X_{i_0}^{\mathfrak{A}} = \emptyset$), thus the sequence $(X_i^{\mathfrak{A}})_{i\geq 0}$ becomes periodic or remains empty.
Anyhow we have $P_\infty^{\mathfrak{A}} = X_\infty^{\mathfrak{A}}$ thus $\mathfrak{A} \models \xi$ if and only if $\mathfrak{A} \in \mathcal{P}$.

Similarly, we can define a class $\mathcal{I}$ of τ–structures that is complete for **IFP** under quantifier–free reductions.
Therefore we first define for each τ–structure $\mathfrak{A}$ a sequence $(I_i^{\mathfrak{A}})_{i\geq 1}$ which is inductive (i.e. $I_i^{\mathfrak{A}} \subseteq I_{i+1}^{\mathfrak{A}}$ for each $i \geq 1$) as follows: We let $I_1^{\mathfrak{A}} = L^{\mathfrak{A}}$ and

$$I_{i+1}^{\mathfrak{A}} = I_i^{\mathfrak{A}} \cup \{a \in A \mid \exists b \in I_i^{\mathfrak{A}}, c \notin I_i^{\mathfrak{A}} : T^{\mathfrak{A}}abc\}$$

for each $i \geq 1$. Here we simply let $I_\infty^{\mathfrak{A}} = \bigcup_{i\geq 1} I_i^{\mathfrak{A}}$.

$\mathcal{I}$ is the class of τ–structures with $R^{\mathfrak{A}} \subseteq I_\infty^{\mathfrak{A}}$.
Analogously to the **PFP**–case it can be seen that $\mathcal{I}$ is complete for **IFP** under quantifier–free reductions.

Hence Theorem 1.1 is proved. ■

4.2 Remark : Our theorem implies that there is a quantifier–free reduction from $\mathcal{P}$ to $\mathcal{I}$ if and only if **IFP** equals **PFP**. By a theorem of Abiteboul and Vianu [AV91b] this is the case if and only if **PTIME**=**PSPACE**.
This is interesting in so far as it gives a combinatorial characterization (that does not need any notion of computation and very few logic) of the question wether **PTIME** equals **PSPACE**. □

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