Some remarks on finite Löwenheim-Skolem theorems

Martin Grohe*

September 1995†

Abstract. We discuss several possible extensions of the classical Löwenheim-Skolem Theorem to finite structures and give a counterexample refuting almost all of them.

Mathematics Subject Classification: 03C13, 03C95

Keywords: Finite Model Theory, Löwenheim-Skolem Theorem

It is known that most theorems from classical model theory, such as the compactness or interpolation theorem, fail when restricted to finite structures (see e.g. [Gur84]). The situation is a bit different for the classical Löwenheim-Skolem theorems: they speak about infinite cardinalities, so their statements do not even make sense when restricted to the finite. However, there are possible formulations of Löwenheim-Skolem theorems for finite structures. The purpose of this note is to discuss some of these.

For example, the following almost trivial statement can be seen as "finite Hanf theorem":

> *For each finite signature σ there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that any first–order sentence of quantifier–rank q which has a model in each cardinality $< f(q)$ has a model in every finite cardinality.*

Similarly we can formulate a "finite Löwenheim theorem".

However, such observations do not help much if we cannot put some restrictions on the function f. At least we would expect f to be recursive. As we will see, the statements do not hold for recursive f.

But there are other possible formulations of Löwenheim-Skolem-type theorems. First of all, we need not ask for a model in *each* cardinality, it would still be nice to have arbitrary large models. Second, if we do not obtain a positive result for whole first–order logic, then we can try at least for some fragments. And third, the quantifier rank of a formula is not the only parameter that can

†To Appear in *Mathematical Logic Quarterly*

*I would like to thank Kerkko Luosto and Lauri Hella for helpful discussions and comments on the subject.

be considered, the function f may throughout depend on other properties of a formula.

Before we give a counterexample which destroys most hopes for such results let us mention a positive result for a fragment of first–order logic which turns out to be optimal. It is quite obvious that our problem is linked to the decision problem of finite satisfiability and thus to the classical "Entscheidungsproblem". In fact the following result was Ramsey's original argument showing that the so called "Bernay–Schönfinkel class with equality" is solvable:

Fact [Ram30]. *For each signature σ consisting only of finitely many relation and constant symbols there exists a (primitive) recursive function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any σ–sentence φ of the form $\exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_l \theta$ where θ is quantifier–free we have:*

If φ has a model in some (even infinite) cardinality of size $\geq f(k,l)$ then it has a model in each cardinality of size $\geq f(k,l)$.

A counterexample

Let $\sigma_{Ar} = \{+, \times, <, S, \underline{0}\}$ be a relational signature of arithmetic, i.e. $+$ and $\times$ are ternary relation symbols, $<$ and S are binary relation symbols and $\underline{0}$ is a constant symbol. Furthermore, let $\mathfrak{N}$ be the standard model of arithmetic, i.e. the σ_{Ar}–structure with universe $\mathbb{N}$ where all symbols are interpreted in the usual way.

Let us fix some "reasonable" Gödel numbering of all first–order σ_{Ar}–formulae. It will become clear from the proof of the proposition what makes a Gödel numbering reasonable. We denote the number of a formula φ by $\ulcorner\varphi\urcorner$ and the formula with number n by ψ_n.

Proposition 1. *For each recursive function f there exists a Π_2–sentence φ of signature σ_{Ar} and an $n_0 > f(\ulcorner\varphi\urcorner)$ such that φ has a model in each cardinality $< n_0$ but no model of cardinality $\geq n_0$.*

Proof: Note first that the class of substructures of $\mathfrak{N}$ whose universe is a finite initial segment of $(\mathbb{N}, <^{\mathfrak{N}})$ can be finitely axiomatised (as a class of finite structures) by the following sentences:

$$
\begin{array}{ll}
(1) & \forall x\, \neg x < x \\
(2) & \forall x \forall y (x < y \vee x = y \vee y < x) \\
(3) & \forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z) \\
(4) & \forall x\, \neg x < \underline{0} \\
(5) & \forall x (\exists y\, Sxy \;\vee\; \forall y (y < x \vee y = x)) \\
(6) & \forall x \forall y (Sxy \rightarrow (x < y \wedge \forall z \neg (x < z \wedge z < y))) \\
(7) & \forall x \forall y \forall z \forall z' ((x + y = z \wedge x + y = z') \rightarrow z = z') \\
(8) & \forall x\, x + \underline{0} = x \\
(9) & \forall x \forall y \forall y' \forall z \forall z' ((x + y = z \wedge Syy' \wedge Szz') \rightarrow x + y' = z') \\
(10) & \forall x \forall y \forall z \forall z' ((x \times y = z \wedge x \times y = z') \rightarrow z = z') \\
(11) & \forall x\, x \times \underline{0} = \underline{0} \\
(12) & \forall x \forall y \forall y' \forall z \forall z' ((x \times y = z \wedge Syy' \wedge z + x = z') \rightarrow x \times y' = z')
\end{array}
$$

Let φ_{Ar} denote the conjunction of all these axioms. φ_{Ar} is a Π_2–sentence, but note that actually all axioms but (5) are universal.

Let us introduce some more notation. For a formula ψ and an $n \in \mathbb{N}$ we let $\psi\frac{n}{x}$ abbreviate the formula

$$\exists x_1 \dots \exists x_{n-1} (S\underline{0}x_1 \wedge \bigwedge_{1 \le i < n-1} Sx_i x_{i+1} \wedge Sx_{n-1}x) \rightarrow \psi.$$

Note that if ψ is universal $\psi\frac{n}{x}$ is also universal.

Let f be a recursive function. By choosing a "reasonable" Gödel numbering we have guaranteed that the function g defined by $g(n) = f(\ulcorner \varphi_{Ar} \wedge \psi_n \frac{n}{x} \urcorner)$ is recursive. So by the well–known Matijasevič-Robinson-Davis-Putnam theorem [Mat70] there exists a quantifier–free σ_{Ar}–formula $\theta(x, y, \bar{z})$ such that for all $m, n \in \mathbb{N}$ we have

$$g(m) = n \quad \Longleftrightarrow \quad \mathfrak{N} \models \exists \bar{z} \theta(m, n, \bar{z}).$$

(It is no problem here that we only have a relational signature; we can compensate this by additional existential quantifiers.)

Let $m_0 = \ulcorner \forall y \forall \bar{z} \neg \theta(x, y, \bar{z}) \urcorner$. Select $n_0 > g(m_0)$ minimal such that there exists a tuple $\bar{l} < n_0$ such that $\mathfrak{N} \models \theta(m_0, g(m_0), \bar{l})$.

Let $\varphi = \varphi_{Ar} \wedge \big(\forall y \forall \bar{z} \neg \theta(x, y, \bar{z})\big)\frac{m_0}{x}$. Suppose for contradiction that φ has a finite model of size $\ge n_0$. Then there exists an $n \ge n_0$ such that the substructure of $\mathfrak{N}$ with universe $\{0, \dots, n-1\}$ is a model of $\forall y \forall \bar{z} \neg \theta(m_0, y, \bar{z})$. However, this contradicts $\mathfrak{N} \models \theta(m_0, g(m_0), \bar{l})$.

On the other hand, by the definition of the substitution of x by m_0 each substructure of $\mathfrak{N}$ with universe $\{0, \dots, n-1\}$ where $0 \le n \le m_0$ is a model of φ, and by the minimality of n_0 each substructure of $\mathfrak{N}$ with universe $\{0, \dots, n-1\}$ where $m_0 < n < n_0$ is a model of φ.

Furthermore, we have

$$f(\ulcorner \varphi \urcorner) = f(\ulcorner \varphi_{Ar} \wedge \big(\forall y \forall \bar{z} \neg \theta(x, y, \bar{z})\big)\tfrac{m_0}{x} \urcorner) = f(\ulcorner \varphi_{Ar} \wedge \psi_{m_0} \tfrac{m_0}{x} \urcorner) = g(m_0) < n_0,$$

completing the proof. ■

Note that the only place where we actually needed φ to be Π_2 was axiom (5) in φ_{Ar}. If a unary function symbol s is added to σ_{Ar} the proof also works for a universal φ — we simply replace (5) by $\forall x \forall y(s(x) = y \rightarrow (Sxy \vee \forall z(z = x \vee z < x)))$. In particular this shows that Ramsey's result mentioned above fails for signatures containing a function symbol.

Also note that the application of the MRDP–theorem is not essential for the proof of Proposition 1 if one is not interested in the prefix of the formula φ. Other descriptions of recursive functions (or Turing machines) by formulae provide similar proofs (using the same diagonalization argument).

We close this note with an observation which can be proved completely analogously to Proposition 1. It again stresses the connection to the decision problem of finite satisfiability (implying the well–known fact that the finite satisfiability of Π_2–sentences is not decidable):

Proposition 2 *For each recursive function f there exists a Π_2–sentence χ of signature σ_{Ar} and an $n_1 > f(\ulcorner \chi \urcorner)$ such that χ has a model in each cardinality $\geq n_1$ but no model of cardinality $< n_1$.*

References

[Gur84] Y. Gurevich. Toward logic tailored for computational complexity. In M.M. Richter et al., editor, *Computation and Proof Theory*, volume 1104 of *Lecture Notes in Mathematics*, pages 175–216. Springer-Verlag, 1984.

[Mat70] J.V. Matijasevič. Enumerable sets are diophantic. *Soviet Mathematics Doklady*, 11:345–357, 1970.

[Ram30] F.P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society, Second Series*, 30:264–286, 1930.

Martin Grohe
Institut für mathematische Logik
Eckerstr. 1
79104 Freiburg
Germany

Email: grohe@sun1.mathematik.uni-freiburg.de