A Simple Algorithm for the Graph Minor Decomposition – Logic meets Structural Graph Theory–

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Abstract

A key result of Robertson and Seymour's graph minor theory is a structure theorem stating that all graphs excluding some fixed graph as a minor have a tree decomposition into pieces that are almost embeddable in a fixed surface. Most algorithmic applications of graph minor theory rely on an algorithmic version of this result. However, the known algorithms for computing such graph minor decompositions heavily rely on the very long and complicated proofs of the existence of such decompositions, essentially they retrace these proofs and show that all steps are algorithmic.

In this paper, we give a simple quadratic time algorithm for computing graph minor decompositions. The best previously known algorithm due to Kawarabayashi and Wollan runs in cubic time and is far more complicated.

Our algorithm combines techniques from logic and structural graph theory, or more precisely, a variant of Courcelle's Theorem stating that monadic second-order logic formulas can be evaluated in linear time on graphs of bounded tree width and Robertson and Seymour's so called Weak Structure Theorem.

1 Introduction

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. The theory of graph minors was developed by Robertson and Seymour in a series of 23 papers published over more than twenty-five years. The aim of that series of papers was the proof of a single result: the *graph minor theorem*, which says that in any infinite collection of finite graphs there is one that is a minor of another. As with other deep results in mathematics, the body of theory developed for the proof of the graph minor theorem has also found applications elsewhere, both within graph theory and computer science. Most of these applications rely not only on the general techniques developed by Robertson and Seymour to handle graph minors, but also on one particular structural result (proved in [29]), which is central to the whole theory. It describes the structure of all graphs G which do not contain some fixed graph H as a minor. At a high level, the theorem says that every such a graph can be decomposed into a collection of graphs each of which can be "nearly" embedded into a bounded-genus surface; the pieces can be assembled in a tree structure to obtain the original graph. In the following, we refer to such a decomposition as a *graph minor decomposition*.

Starting with Robertson and Seymour's cubic time algorithm for the disjoint path problem [28], a substantial body of work on "algorithmic graph minor theory" emerged (e.g. [7, 8, 9, 14, 15, 17]). Most of the results give efficient (exact or approximation) algorithms for various hard problems on classes of graphs with excluded minors, but some go beyond such classes [28, 15]. Almost all of these results rely, either directly or indirectly through other results, on the existence of graph minor decompositions (i.e., Robertson and Seymour's structure theorem) and on *efficient algorithms for computing these decompositions*.

Several such algorithms are known. The third author of this paper was maybe the first to point out that Robertson-Seymour's original proof of the structure theorem (which requires almost 400 pages) gives rise to a polynomial time algorithm to construct the decompositions. Demaine, Hajiaghayi, and Kawarabayashi [9] give a lengthy proof for constructing it, which builds on many structural graph minor results. The running time of this algorithm is $O(n^k)$ for a k that depends on the size of the excluded minor. Dawar, Grohe, and Kreutzer [6] give an fixed-parameter algorithm that, however, computes a "weaker" decomposition into pieces that have bounded local tree width

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after removing a bounded number of vertices. Recently, Kawarabayashi and Wollan [18] found a dramatically shorter proof for the graph minor decomposition theorem (cutting off around 300 pages of the original graph minor papers), which yields a cubic time algorithm for computing graph minor decompositions. However, all these algorithms are deeply entrenched in rather heavy structural graph theory. Essentially, they are algorithmic proofs of the structure theorem.

In this paper, we give a simple quadratic time algorithm to construct a graph minor decomposition. We take a completely different approach than the previous algorithms. We take the existence of a decomposition for granted and just try to find one. We reduce structural graph theory to a minimum, but combine it with tools from logic. The correctness proof of our algorithm essentially fits within this conference paper,[1] which is quite remarkable when compared with previous algorithms.

Our main technical contributions are the following.

(A) We prove one graph theoretic result, which roughly says that if there is a vertex v deep inside a grid in a graph G, then no matter how we obtain a graph minor decomposition of $G - v$, we can put v back into this decomposition and thus obtain a decomposition of G.

(B) We prove that "near embeddings" (in the sense required for graph minor decompositions) of graphs in a bounded genus surface are definable in monadic second logic.

Using these two results, our algorithm proceeds as follows. Repeatedly applying (A), it deletes vertices from the input graph G until it arrives at a graph G' that no longer has a large grid. By the Excluded Grid Theorem [25], G' has bounded tree width. Applying an extension of Courcelle's well known theorem [2, 4], stating that monadic second-order formulas can be evaluated in linear time, to the formulas obtained in (B), we obtain a linear time algorithm for constructing near embeddings of bounded tree width graphs. We can even construct such near embeddings "locally" with respect to a tangle. (A *tangle* [27] may be viewed as a structure describing a highly connected region in a graph, and a near embedding is *local* with respect to the tangle if the part of the graph properly embedded falls within this region.) Now we can use a (fairly simple and by now standard) generic construction going back to [27] for computing a "global" tree decomposition respecting the local structure, in our case near embeddability. This enables us to compute a graph minor decomposition of the bounded-tree-width graph G' in quadratic time. Then, in the third step, our algorithm re-inserts the vertices deleted in the first step and extends the decomposition from G' to G.

The paper is organised as follows: After general preliminaries in Section 2, we formally define graph minor decompositions in Section 3 and state the structure theorem. We prove claim (A) from above in Section 4 and claim (B) in Section 5. We put everything together in Section 6.

2 Preliminaries

For all integers m, n, we denote the set $\{m, m+1, \ldots, n\}$, which is empty if $m > n$, by $[m, n]$, and we let $[n] := [1, n]$. We use a standard graph theoretic terminology and notation. The set of all neighbours of a vertex w or a set W of vertices in a graph G is denoted by $N^G(w)$ and $N^G(W)$, respectively, and for a subgraph H of G we let $N^G(H) = N^G(V(H))$.

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.

(2) If $t, t', t'' \in V(T)$ and t' lies on the path in T from t to t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

For every node $t \in V(T)$, we call Y_t the *bag* at t. The *torso* at t is the graph H_t obtained from the induced subgraph $G[Y_t]$ by adding edges between all vertices v, w such that $v, w \in Y_t \cap Y_u$ for some neighbour $u \in N^T(t)$.

It is sometimes convenient to view the tree in a tree decomposition as rooted, and will freely do so.

The *adhesion* of a tree decomposition (T, Y) is $\max\{|Y_t \cap Y_u| \mid tu \in E(T)\}$ if $E(T) \neq \emptyset$, and 0 if $E(T) = \emptyset$. The *width* of (T, Y) is $\max\{|Y_t| \mid t \in V(T)\} - 1$, and the *tree width* $\mathrm{tw}(G)$ of G is defined as the minimum width taken over all tree decompositions of G. By a well-known result due to Bodlaender [3], there is an algorithm that, given a graph G and an integer w, decides if the tree width of G is at most w and computes a tree decomposition of G of width at most w if it is. The running time of the algorithm is $2^{O(w^3)}n$, where $n = |G|$.

3 Structure Theorems

3.1 Tangles Let G be a graph. A *separation* of G is a pair (A, B) of subgraphs such that $G = A \cup B$ (hence there are no edges between $A - B$ and $B - A$). The order of the separation (A, B) is $|A \cap B|$. A *tangle* of order k of G is a set $\mathfrak{T}$ of separations of G of order $< k$

[1] Some proofs in this conference paper are only sketched.

satisfying the following there conditions.

(1) For all separations (A, B) of G of order $< k$, either $(A, B) \in \mathfrak{T}$ or $(B, A) \in \mathfrak{T}$.

(2) If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathfrak{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$.

(3) If $(A, B) \in \mathfrak{T}$ then $V(A) \neq V(G)$.

Note that if $(A, B) \in \mathfrak{T}$ then $(B, A) \notin \mathfrak{T}$; we think of B as the 'big side' of the separation (A, B), with respect to this tangle.

A separation (A, B) of G *breaks* a set $U \subseteq V(G)$ if $|(V(A) \cap U) \cup V(A \cap B)| < |U|$ and $|(V(B) \cap U) \cup V(A \cap B)| < |U|$. The set U is *k-unbreakable* if there is no separation (A, B) of G of order $< k$ that breaks U.

For a k-unbreakable set U of size $|U| \geq 3k - 2$ we define $\mathfrak{T}_U$ to be the set of all separations of (A, B) of G of order at most k such that $|(V(B) \cap U) \cup V(A \cap B)| \geq |U|$. Then $\mathfrak{T}_U$ is a tangle of order k of G [12, 15, 27].

We close this short introduction to tangles with a characterisation of tangles in terms of separators and components rather than separations.

Lemma 3.1. ([24]) *Let $\mathfrak{T}$ be a tangle of order k in a graph G. Then for every $S \subseteq V(G)$ with $|S| < k$ there is a unique connected component $C_{\mathfrak{T},S}$ of $G \setminus S$ such that for all separations (A, B) of G with $V(A \cap B) = S$ we have $(A, B) \in \mathfrak{T} \iff C_{\mathfrak{T},S} \subseteq B$.*

3.2 Societies and Vortices A *society* is a pair (G, Ω), where G is a graph and Ω a cyclic permutation of a subset $V(\Omega)$ of $V(G)$. The vertices in $V(\Omega)$ are sometimes called *society vertices*. Note that for every $w \in V(\Omega)$ we have $V(\Omega) = \{\Omega^j(w) \mid j \in [0, |V(\Omega)| - 1]\}$. The *length* of a society (G, Ω) is $|V(\Omega)|$.

A society (G, Ω) of length ℓ is a *α-vortex* if for all $w \in V(\Omega)$ and $k \in [\ell]$ there do not exist $(\alpha + 1)$ mutually disjoint paths of G between $\{\Omega^j(w) \mid j \in [0, k-1]\}$ and $\{\Omega^j(w) \mid j \in [k, \ell - 1]\}$.

A *linear decomposition* of a society (G, Ω) of length ℓ is a sequence $(X_i)_{i \in [0, \ell-1]}$ of subsets of $V(G)$ such that

(1) $\bigcup_{i=0}^{\ell-1} X_i = V(G)$

(2) $X_i \cap X_k \subseteq X_j$ for all $i, j, k \in [0, \ell - 1]$ with $i \leq j \leq k$.

(3) There is a $x_0 \in V(\Omega)$ such that $\Omega^i(x_0) \in X_i$ for all $i \in [0, \ell - 1]$.

The *width* of the linear decomposition $(X_i)_{i \in [0,\ell-1]}$ is $\max\{|X_i| \mid i \in [0, \ell - 1]\}$, and the *depth* of $(X_i)_{i \in [0,\ell-1]}$ is $\max\{|X_i \cap X_{i+1}| \mid i \in [0, \ell - 1]\}$. Sometimes X_i is called a *bag* of the linear decomposition.

It is proved in [26] that if a society (G, Ω) is a α-vortex then it has a linear decomposition of depth at most α.

Lemma 3.2. *For all nonnegative integers α, k there is a linear time algorithm that, given a an α-vortex (G, Ω) such that $\mathrm{tw}(G) \leq k$, computes a linear decomposition of (G, Ω) of depth at most $2\alpha + 2$.*

Proof. Let (G, Ω) be an α-vortex such that $\mathrm{tw}(G) \leq k$. We choose an arbitrary $w_0 \in V(\Omega)$ and let $w_i = \Omega^i(w_0)$ for all $i \geq 1$. Observe that (G, Ω) has a linear decomposition $(X_i)_{0 \leq i < \ell}$ of depth at most 2α such that $w_i \in X_i$ for all $i \in [0, \ell - 1]$. To see this, take a linear decomposition $(X'_i)_{i \in [0,\ell-1]}$ of (G, Ω) of depth at most α. Such a decomposition exists by the result of [26]. Then for some $j \in [0, \ell - 1]$ we have $w_{i+j} \in X'_i$ for all $i \in [0, \ell - 1]$. If $j = 0$, we simply let $X_i = X'_i$ for all i. If $j > 0$ we let $Z = X_{\ell-j-1} \cap X_{\ell-j}$ and $X_i = X'_{i+\ell-j} \cup Z$ for all i, where the indices are taken modulo ℓ. It is easy to verify that $(X_i)_{0 \leq i < \ell}$ is a linear decomposition of (G, Ω) with the desired properties.

Now we let $q = k + 2\alpha + 1$. We let H be the graph obtained from G by adding

- fresh vertices w_{ij} for $i \in [0, \ell - 1]$, $j \in [q - 1]$,
- edges $w_i w_{ij}$ for $i \in [0, \ell - 1]$, $j \in [q - 1]$,
- edges $w_{ij} w_{ij'}$ for $i \in [0, \ell - 1]$, $j \neq j' \in [q - 1]$,
- edges $w_i w_{i+1}$ for $i \in [0, \ell - 1]$ if they are not present in G,
- edges $w_{ij} w_{(i+1)j'}$ for $i \in [0, \ell - 1]$, $j, j' \in [q - 1]$.

Then for all i, the set $W_i = \{w_i\} \cup \{w_{ij} \mid 1 \leq j \leq q - 1\}$ is a q-clique in G, and for $i \in [0, \ell - 2]$ the set $W_i \cup W_{i+1}$ is a $2q$-clique.

We claim that the tree width of H is at most $2q + 2\alpha - 1$. To prove this claim, we construct a tree decomposition (T, Y) of H of that width. We start with the linear decomposition $(X_i)_{i \in [0,\ell-1]}$. To simplify the notation, let $X_{-1} = \emptyset$. For $i \in [0, \ell - 1]$, we let $Y_{2i} = (X_{i-1} \cap X_i) \cup (X_i \cap X_{i+1}) \cup W_i$, and if $i < \ell - 1$ we let $Y_{2i+1} = (X_i \cap X_{i+1}) \cup W_i \cup W_{i+1}$. Note that for all $i \in [0, 2\ell - 2]$ we have $|Y_i| \leq 2q + 2\alpha$, because $|X_{i-1} \cap X_i| \leq 2\alpha$ and $q \geq 2\alpha$. Let P be the path with vertices $0, \ldots, 2\ell - 2$ and edges $i(i+1)$ for $i \in [0, 2\ell - 3]$. For $i \in [0, \ell - 1]$, let (T^i, Y^i) be a tree decomposition of the graph $G[X_i]$ of width $2q + 2\alpha - 1$ such that the set $(X_{i-1} \cap X_i) \cup (X_i \cap X_{i+1}) \cup \{w_i\}$ is contained in the bag $Y^i_{r^i}$ of the root r^i of T^i. Such a tree decomposition exists because we can take a tree decomposition of $G[X_i]$ of width k and add the set $(X_{i-1} \cap X_i) \cup (X_i \cap X_{i+1}) \cup \{w_i\}$ to every bag. As $k + 1 + 4\alpha + 1 \leq 2q + 2\alpha$, we have $|Y^i_t| \leq 2q + 2\alpha$ for all bags Y^i_t of this tree decomposition. Without loss of generality we may assume that the trees T^i are mutually disjoint and disjoint from the path P. We form a tree T by taking the union of the P and the trees T^i and for $i \in [0, \ell-1]$ adding an edge from node $2i$ of P to the root r^i of T^i. For $i \in V(P)$, we define Y_i as above, and for $t \in V(T^i)$, we let $Y_t = Y^i_t$. Then (T, Y) is a tree decomposition of H of width at most $2q + 2\alpha - 1$.

The crucial point is that we can revert the construction: from every tree decomposition of H of width at most $2q + 2\alpha - 1$ we can construct a linear decomposition of (G, Ω) of depth at most $2\alpha + 1$. Let (T, Y) be a tree decomposition of H of width at most $2q + 2\alpha - 1$.

For $i \in [0, \ell - 2]$, let U_i be the set of all nodes $u \in V(T)$ such that $W_i \cup W_{i+1} \subseteq Y_u$. Then $U_i \neq \emptyset$, because $W_i \cup W_{i+1}$ is a clique in H, and U_i is connected in T. For $i \neq j$ we have $U_i \cap U_j = \emptyset$, because if $u \in U_i \cap U_j$ then Y_u contains at least three of the sets W_k, which means $|Y_u| \geq 3q > 2q + 2\alpha$. This contradicts the width of (T, Y) being at most $2q + 2\alpha - 1$. We construct a path $P \subseteq T$ and nodes $t(0), t(1), \ldots, t(\ell-2)$ appearing on P in the right order such that $t(i) \in U_i$ for all i. We let $t(0) \in U_0$ be arbitrary. Suppose that for some $i < \ell - 2$ we have defined $t(0), \ldots, t(i)$ and the segment P_i of P from $t(0)$ to $t(i)$. Let $t(i + 1)$ be the vertex in U_{i+1} closest to $t(i)$, and let Q be the path from $t(i)$ to $t(i + 1)$ in T. We claim that Q and P_i are internally disjoint. If not, $t(i)$ has the same neighbour, say t, on P_i and Q. We have $W_i \subseteq Y_t$, because $W_i \subseteq Y_{t(i-1)} \cap Y_{t(i)}$ and $W_{i+1} \subseteq Y_t$, because $W_{i+1} \subseteq Y_{t(i)} \cap Y_{t(i+1)}$. Thus $t \in U_i$, and t is closer to $t(i - 1)$ than $t(i)$. This contradicts our construction. Hence P_i and Q must be internally disjoint, and we let $P_{i+1} = P_i \cup Q$.

Now we are ready to construct the desired linear decomposition of G. For $i \in [\ell - 2]$, we let

$$Z_i = (Y_{t(i)} \setminus (W_i \cup W_{i+1})) \cup \{w_i, w_{i+1}\}.$$

Note that $|Z_i| \leq |Y_{t(i)}| - 2q + 2 \leq 2\alpha + 2$. We let X_0 be the union of Z_0 with all sets $Y_t \cap V(G)$ such that t is in a connected component of $T \setminus \{t(1)\}$ that has an empty intersection with P. For $2 \leq i \leq \ell - 2$, we let X_i be the union of $Z_{i-1} \cup Z_i$ with all sets $Y_t \cap V(G)$ such that t is in the connected component of $T \setminus \{t(i - 1), t(i)\}$ that contains the segment of P from $t(i - 1)$ to $t(i)$. We let $X_{\ell-1}$ be the union of $Z_{\ell-2}$ with all sets $Y_t \cap V(G)$ such that t is in a connected component of $T \setminus \{t(\ell-2)\}$ that has an empty intersection with P. It is easy to see that $(X_i)_{0 \leq i < \ell}$ is a linear decomposition of (G, Ω), and as $X_i \cap X_{i+1} = Z_i$ for all i, the depth of this decomposition is at most $2\alpha + 2$.

Our algorithm for computing such a linear decomposition proceeds as follows: On input (G, Ω), it first constructs the graph H. Then it uses Bodlaender's linear time algorithm to compute a tree decomposition (T, Y) of H of width at most $2q + 2\alpha - 1$, and from this tree decomposition it computes the desired linear decomposition $(X_i)_{i \in [\ell-1]}$ of (G, Ω) following the construction described above. □

3.3 Near Embeddings For nonnegative integers $\alpha_1, \alpha_2, \alpha_3$, a graph G is $(\alpha_1, \alpha_2, \alpha_3)$-*nearly embeddable* in a surface Σ if there is a subset $Z \subseteq V(G)$ with $|Z| \leq \alpha_1$, two sets $\mathcal{V} = \{(G_1, \Omega_1), \ldots, (G_{\alpha_2}, \Omega_{\alpha_2})\}$ and $\mathcal{W} = \{(G_{\alpha_2+1}, \Omega_{\alpha_2+1}), \ldots, (G_n, \Omega_n)\}$ of societies, and a graph G_0 such that the following conditions are satisfied.

(1) $G \setminus Z = G_0 \cup G_1 \cup \ldots \cup G_n$.

(2) For all $i \in [n]$ we have $E(G_0) \cap E(G_i) = \emptyset$ and $V(G_0) \cap V(G_i) \subseteq V(\Omega_i)$.
For all distinct $i, j \in [n]$ we have $E(G_i) \cap E(G_j) = \emptyset$ and $V(G_i) \cap V(G_j) \subseteq V(\Omega_i) \cap V(\Omega_j)$. Furthermore, if $i, j \leq \alpha_2$ then $V(G_i) \cap V(G_j) = \emptyset$.

(3) Each $(G_i, \Omega) \in \mathcal{V}$ is an α_3-vortex. (By a result of [26], (G_i, Ω_i) has a linear decomposition of depth at most α_3.)

(4) Each $(G_i, \Omega_i) \in \mathcal{W}$ has length at most 3.

(5) There are closed disks $\Delta_1, \ldots, \Delta_n \subseteq \Sigma$ with disjoint interiors and an embedding $\sigma : G_0 \hookrightarrow \Sigma$ such that for all $i \in [n]$ we have $\sigma(G_0) \cap \mathrm{int}(\Delta_i) = \emptyset$ and $\sigma(V(G_0)) \cap \mathrm{bd}(\Delta_i) = \sigma(V(\Omega_i))$, and the cyclic ordering of the vertices in $\sigma(V(\Omega_i))$ induced by Ω_i is compatible with the natural cyclic ordering of the vertices on the simple closed curve $\mathrm{bd}(\Delta_i)$. We say that the disk Δ_i is *accommodating* (G_i, Ω_i).

We call $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ an $(\alpha_1, \alpha_2, \alpha_3)$-*near embedding* of G in Σ or just *near-embedding* if the parameters are clear from the context. For a nonnegative integer α, an α-*near embedding* is an $(\alpha_1, \alpha_2, \alpha_3)$-near embedding where $\alpha_1, \alpha_2, \alpha_3 \leq \alpha$.

Let G'_0 be the graph resulting from G_0 by joining any two nonadjacent vertices $u, v \in G_0$ such that $u, v \in \Omega$ for some society $(J, \Omega) \in \mathcal{W}$; the new edge uv of G'_0 will be called a *virtual edge*. By embedding these virtual edges disjointly in the disks Δ accommodating their societies, we extend our embedding $\sigma \colon G_0 \hookrightarrow \Sigma$ to an embedding $\sigma' \colon G'_0 \hookrightarrow \Sigma$. We shall not normally distinguish G'_0 from its image in Σ under σ'. Observe that if $C \subseteq G$ is a cycle with $C \cap G_i = \emptyset$ for all $i \in [\alpha_2]$ there is a unique cycle $C' \subseteq G'_0$ obtained from C by replacing segments in graph G_i for $i > \alpha_2$ by virtual edges. We call C' the *shortcut for C in G'_0*.

A near-embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ is *nice* if for all $(J, \Omega) \in \mathcal{V}$ there is a cycle $C \subseteq G'_0$ such that C is the boundary of the disk Δ accommodating (J, Ω).

A near-embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ is $\mathfrak{T}$-*central*, for a tangle $\mathfrak{T}$ of G, if for all $(J, \Omega) \in \mathcal{V} \cup \mathcal{W}$ there is no $(A, B) \in \mathfrak{T}$ with $Z \subseteq V(A \cap B)$ and $B \setminus Z \subseteq J$.

The following theorem may be viewed as the main structural result of graph minor theory. We state mild a strengthening of Robertson and Seymour's original theorem ([29], Theorem (3.1)) due to [12] in that we require the near embedding to be nice. (Note that working with this strengthening makes our algorithmic results stronger as well, because we will obtain *nice* near embeddings for the pieces of the decomposition we shall compute.)

Theorem 3.1. (Local Structure Theorem) *For every graph R there are nonnegative α, β, γ such that the following holds. Let G be a graph that excludes R as a minor and $\mathfrak{T}$ a tangle of G of order at least β. Then G has a $\mathfrak{T}$-central nice α-near embedding in a surface of Euler genus at most γ.*

This theorem follows easily from Theorem 2 of [12], noting that $(3,3)$-rich near embeddings are nice and that if we choose β sufficiently large, every graph of small tree width does not have a tangle of order at least β.

Let us call a graph G *locally* $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-*decomposable* if for every β-unbreakable set $U \subseteq V(G)$ of size $|U| = 3\beta - 2$ the graph has a $\mathfrak{T}_U$-central nice $(\alpha_1, \alpha_2', \alpha_3)$-near embedding in a surface of Euler genus at most γ, for some $\alpha_2' \leq \alpha_2$. It follows from the Local Structure Theorem that for every graph H there are parameters $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ such that for all $\beta' \geq \beta$, all H-minor-free graphs are locally $(\alpha_1, \alpha_2, \alpha_3, \beta', \gamma)$-decomposable.

The following theorem, which is a strengthening of Robertson and Seymour's structure theorem from [29], follows from Theorem 3.1 by standard techniques; see [12] for a proof. An $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-*decomposition of* G is a tree decomposition (T, Y) of G, where we view T as a rooted tree, such that for all nodes t, either $|Y_t| \leq 4\beta - 3$ or for some $\alpha_2' \leq \alpha_2$ the torso H_t has a nice $(\alpha_1, \alpha_2', \alpha_3)$-near embedding $(\sigma_t, H_{t0}, Z_t, \mathcal{V}_t, \emptyset)$ in a surface Σ_t of Euler genus at most γ where all vortices $(J, \Omega) \in \mathcal{V}$ have a linear decomposition $D(J, \Omega)$ of depth at most α_3 and width at most $2\alpha_3 + 1$. If $|Y_t| \leq 4\beta - 3$, we call t *small*; otherwise, we call t a *nearly embeddable*. All nearly embeddable nodes t satisfy the following additional conditions. For all children u of t in T,

- either $Y_u \cap Y_t \subseteq V(H_{t0}) \cup Z_t$ and $|(Y_u \cap Y_t) - Z_t| \leq 3$ and there is a closed disk $\Delta \subseteq \Sigma_t$ such that $\sigma_t(H_{t0}) \cap \text{int}(\Delta) = \emptyset$ and $\sigma_t(H_{t0}) \cap \text{bd}(\Delta) = \sigma_t((Y_u \cap Y_t) - Z_t)$,
- or there is a vortex $(J, \Omega) \in \mathcal{V}_t$ such that $(Y_u \cap Y_t) - Z_t$ is contained in a bag of the linear decomposition $D(J, \Omega)$.

Furthermore, if t is not the root of T, then for the parent s of t we have $Y_s \cap Y_t \subseteq Z_t$.

Note that if $\alpha_1 \geq 4\beta - 3$, we need no small nodes, because for a graph H of order $|H| \leq 4\beta - 3$ we have a trivial near embedding where all vertices are in the set Z.

Theorem 3.2. (Global Structure Theorem) *For every graph R there are nonnegative $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ such that every graph G that excludes R as a minor has an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition.*

Our main algorithm for constructing a global decomposition of a locally decomposable graph can actually be used to prove the Global Structure Theorem from the local one (see Remark 6.1).

Proviso 3.1. *To simplify the presentation, let us agree that whenever we are given an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of a graph in an algorithmic context, we are also given near embeddings of all nearly embeddable nodes of the decomposition, and whenever we have an algorithm computing an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition, the algorithm will also compute appropriate near embeddings.*

4 Location of a wall of large height

An *elementary wall of height* $h \geq 1$ is a graph defined as in Figure 1. A *wall* of height h (or an h-wall) is obtained from an elementary wall of height h by subdividing some of the edges, i.e., replacing the edges with internally vertex disjoint paths with the same endpoints.

The *nails* of a wall are the vertices of degree three within it. Any wall of height $h \geq 3$ has a unique planar embedding where the external face is not a "brick". The boundary cycle of this external face is the *perimeter* of the wall. (The perimeter the unique facial cycle that contains more than 6 nails.) For any wall W in a given graph H, there is a unique component U of $H - \text{per}(W)$ containing $W - \text{per}(W)$. The *compass* of W, denoted $comp_G(W)$, consists of the graph with vertex set $V(U) \cup V(\text{per}(W))$ and edge set $E(U) \cup E(\text{per}(W)) \cup \{xy \mid x \in V(U), y \in V(\text{per}(W))\}$. A *subwall* of a wall W is a wall which is a subgraph of W. A subwall of W of height h is *proper* if it consists of h consecutive bricks from each of h consecutive rows of W.

A *flat wall decomposition* in a graph G is a tuple $(\rho, W, K_0, \ldots, K_n)$, where $W \subseteq G$ is a wall and $K_0, K_1, \ldots, K_n \subseteq G$ are pairwise edge-disjoint subgraphs, and ρ is an embedding of K_0 in the plane such that the following conditions are satisfied.

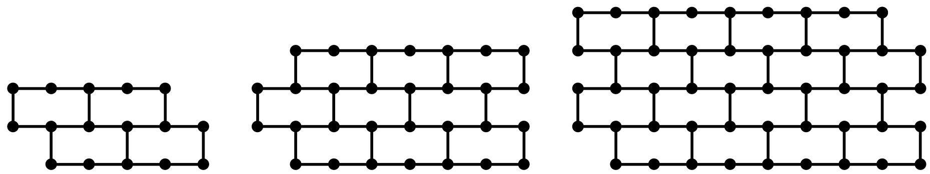

Figure 1: Elementary walls of heights 2–4

(1) $\text{comp}_G(W) = K_0 \cup \ldots, \cup K_n$.

(2) For all distinct $i, j \in [n]$ we have $V(K_i) \cap V(K_j) \subseteq V(K_0)$, and for all $i \in [n]$ we have $|V(K_i) \cap V(K_0)| \leq 3$.

(3) All nails of W are in $V(K_0)$.[2]

(4) There is a closed disk Γ in the plane such that $\rho(K_0) \subseteq \Gamma$ and ρ maps the perimeter cycle of W onto the boundary of Γ. Furthermore, there are closed disks $\Gamma_1, \ldots, \Gamma_n \subseteq \Gamma$ with mutually disjoint interiors such that for $1 \leq i \leq n$ we have $\rho(K_0) \cap \text{int}(\Gamma_i) = \emptyset$ and $\rho(K_0) \cap \text{bd}(\Gamma_i) = V(K_0) \cap V(K_i)$.

Let K_0' be the graph obtained from K_0 by joining any two nonadjacent vertices $u, v \in V(K_0) \cap V(K_i)$, for $1 \leq i \leq n$. We refer to the edges in $E(K_0') \setminus E(K_0)$ as *virtual edges*. We extend the embedding ρ to an embedding ρ' of K_0' in Γ by embedding the new edges in the disks Γ_i. We usually do not distinguish K_0' from its image $\rho'(K_0')$ in Γ.

The *height* of a flat wall decomposition is the height of its wall. A wall W is *flat* in a graph G if there is a flat wall decomposition $(\rho, W, K_0, \ldots, K_n)$ in G. Note that every subwall of a flat wall is flat as well.

Let $(\rho, W, K_0, \ldots, K_n)$ be a flat wall decomposition in a graph G and $K := \text{comp}_G(W) = \bigcup_{i=0}^n K_i$. A cycle $C \subseteq K$ is *flat* if $C \not\subseteq K_i$ for any $i \geq 1$. In particular, every cycle $C \subseteq W$ except possibly a brick in the corner is flat, because it contains at least four nails of W and every K_i for $i \geq 1$ contains at most 3 nails. Observe that for every flat cycle $C \subseteq K$ there is a unique cycle $C' \subseteq K_0$ obtained from C by replacing segments in K_i for $i \geq 1$ by virtual edges. We call C' the *shortcut for C in K_0'*. There is a unique disk $\Gamma(C) \subseteq \Gamma$ such that $\rho'(C') = \text{bd}(\Gamma(C))$; we call $\Gamma(C)$ the disk *bounded* by C (or by C'). An *ℓ-nest* (with respect to $(\rho, W, K_0, \ldots, K_n)$) is a family $C_1, \ldots, C_\ell \subseteq W$ of disjoint flat cycles such that

$$\Gamma(C_1) \subseteq \Gamma(C_2) \subseteq \ldots \subseteq \Gamma(C_\ell).$$

A generic way of constructing an ℓ-nest for $\ell \leq \lceil h/2 \rceil$ is to let C_ℓ be the perimeter of W, $C_{\ell-1}$ the perimeter of the subwall that remains after deleting C_ℓ, et cetera. We call the ℓ-nest constructed this way the *generic ℓ-nest* in W.

Robertson and Seymour's Excluded Grid Theorem [25] states that every graph of sufficiently large tree width (depending on h) contains a wall of height h as a subgraph. The following theorem, also due to Robertson and Seymour [28], is a strengthening of the Excluded Grid Theorem. The linear time algorithm is from [17], which combines Robertson and Seymour's original quadratic time algorithm with Perkovic and Reed's [22] linear time algorithm for computing grids in graphs of large tree width, and an algorithm to compute a flat embedding in linear time.

Theorem 4.1. (Weak Structure Theorem) *For all positive integers m, h there is a positive integers k such that for every graph G of tree width at least k that excludes K_m as a minor, there is a subset $X \subseteq V(G)$ of size $|X| \leq \binom{m}{2}$ and a flat wall of height h in $G \setminus X$.*

Furthermore, there is a linear time algorithm that, given a graph of tree width at least k that excludes K_m as a minor, computes a subset $X \subseteq V(G)$ of size $|X| \leq \binom{m}{2}$ and a flat wall decomposition of height h in $G \setminus X$.

Let W be a wall in graph a G. We view W as embedded in the plane in the natural way. A curve γ in the plane is *W-normal* if it only meets W in vertices. The *face distance (with respect to W)* between two vertices $v_1, v_2 \in V(W)$ is defined to be the minimal value $|V(W) \cap \gamma| - 1$ taken over all curves γ in the plane that link v_1 and v_2. The *face distance (with respect to W in G)* between two vertices $v_1, v_2 \in V(G)$ is 0 if v_1, v_2 belong to the same connected component of $G \setminus V(W)$, and the minimum of the face distances between all $v_1', v_2' \in V(W)$ such that there is a path from $P_i \subseteq G$ from v_i to v_i' with all vertices except v_i' in $V(G) \setminus V(W)$.

[2] In the literature, this condition is usually replaced by the following weaker condition.

(3') For all $i \in [n]$ there is at most one nail of W in $V(K_i) \setminus V(K)$.

However, it is easy to construct a decomposition satisfying (1)–(4) from one that only satisfies (1), (2), (3') and (4): we locally redefine W by moving the nails from the K_i to $K_0 \cap K_i$ and possibly splitting K_i into several components.

A set $X \subseteq V(G)$ is (c,d)-*wide* over a wall $W \subseteq G - X$ if every $x \in X$ has neighbours $v_1, \ldots, v_c \in \text{comp}_{G-X}(W)$ of mutual face distance at least d with respect to W in $G - X$. We need the following corollary of the Weak Structure Theorem.

Corollary 4.1. *For all positive integers c, d, m, h there is a positive integers k such that for every graph G of tree width at least k that excludes K_m as a minor, there is a subset $X \subseteq V(G)$ of size $|X| \le \binom{m}{2}$ and a flat wall of height h in $G \setminus X$ over which X is (c,d)-wide*

Furthermore, there is a linear time algorithm that, given a graph of tree width at least k that excludes K_m as a minor, computes a subset $X \subseteq V(G)$ of size $|X| \le \binom{m}{2}$ and a flat wall decomposition $(\rho, W, K_0, \ldots, K_n)$ of height h in $G \setminus X$ such that X is (c,d)-wide over W.

Let W be a wall in a graph G. A vertex $v \in \text{comp}_G(W)$ is ℓ-*central* in W it its face distance from the perimeter of W is at least ℓ. Observe that if $(\rho, W, K_0, \ldots, K_n)$ is a flat wall decomposition in G and v is ℓ-central in W then there is an ℓ-nest $C_1, \ldots, C_\ell$ such that $\rho(v) \in \text{int}(\Gamma(C_1))$.[3] Indeed, we can take $C_1, \ldots, C_\ell$ to be the generic ℓ-nest in W.

The following lemma is the main result of this section (and the main graph theoretic result of the paper).

Lemma 4.1. (Extension Lemma) *For all nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \xi$ there are nonnegative integers c, d, h, ℓ such that for every graph G the following holds. Suppose that there is a subset $X \subseteq V(G)$ of size $|X| \le \xi$ and a flat wall decomposition $(\rho, W, K_0, K_1, \ldots, K_n)$ in $G \setminus X$ of height at least h such that X is (c,d)-wide over W. Let $w \in V(K_0)$ be ℓ-central in W. Then if $G - w$ has an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition, so does G.*

Furthermore, there is a linear time algorithm that, given G and X and $(\rho, W, K_0, K_1, \ldots, K_n)$ and w and an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of $G - w$, computes an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of G.

The proof of the extension lemma requires some preparation. Let $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \xi$ be nonnegative integers and

$$\alpha = \max\{1, \alpha_1, \alpha_2, \alpha_3, \beta\}.$$

We start with a simple and well-known lemma, whose straightforward proof we omit (cf. [16]).

[3] The converse of this claim is "almost" true; the only thing that may happen is that v belongs to a connected component A of $G - V(W)$ such that $\rho(A) \subseteq \text{int}(\Gamma(C_1))$, but $N(A) \subseteq C_1$, which means that the face distance of v to the perimeter of W may only be $\ell - 1$.

Lemma 4.2. *Let W be a wall of height h in a graph G. Then for every separator $S \subseteq V(G)$ of order $q := |S| < h$ there is exactly one connected component H such that $V(H) \cup S$ contains more than q^2 nails of W.*

We use this to prove the following lemma, which will allow us to focus on a single node of a decomposition.

Lemma 4.3. *Let G be a graph and (T, Y) an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of G. Furthermore, let W be a wall of height $h > 4\alpha$ in G. Then there is a unique node $t \in V(T)$ with the following two properties.*

(1) For each connected component T' of $T - t$ the set $\bigcup_{t' \in V(T')} Y_{t'}$ contains at most $16\alpha^2$ nails of W.

(2) t is a nearly embeddable node.

Proof. It follows easily from the definition of $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decompositions that the adhesion of (T, Y) is at most $q := \max\{\alpha_1 + 2\alpha_3 + 1, 3, 4\beta - 3\} \le 4\alpha$.

For every edge tu of T, let T_{tu} be the component of $T - tu$ containing u, and let $V_{tu} = \bigcup_{t' \in V(T_{tu})} Y_{t'}$. Note that $V_{tu} \cup V_{ut} = V(G)$ and $|V_{tu} \cap V_{ut}| \le q$. Then by Lemma 4.2, either V_{tu} or V_{ut} contains at most q^2 nails of W. Note that it cannot happen that both V_{tu} and V_{ut} contain most q^2 nails, because a wall of height $h > q$ has $2(h-1)(h+1) > 2q^2$ nails.

Let us fix an arbitrary root r for T and then choose a node t such that

- for all directed edges $s't'$ on the path from r to t the set $V_{s't'}$ contains more than q^2 nails of W;
- for all children u of t the set V_{tu} contains at most q^2 nails of W.

Note that if t is not the root and s is its parent, then V_{ts} contains at most q^2 nails of W, because V_{st} contains more than q^2 nails. As each connected component of $T - t$ is of the form T_{tu} for some neighbour u of t, this proves (1). Furthermore, t is unique, because there is no edge tu such that both V_{tu} and V_{ut} contain at most q^2 nails of W.

It remains to prove that t is a nearly embeddable node. Suppose for contradiction that it is a small node, that is, $|Y_t| \le 4\beta - 3 \le q < h$. By Lemma 4.2 and because W has more than $q^2 + q$ nails, there is a connected component of $G \setminus Y_t$ that contains more than q^2 nails. This contradicts (1). $\square$

In the following, we let c, d, h, ℓ be a sufficiently large integers (to be determined later).

Let G be a graph and $X \subseteq V(G)$ with $|X| \le \xi$ and $(\rho, W, K_0, K_1, \ldots, K_n)$ a flat wall decomposition in $G \setminus X$ of height $h \geq 2\ell$ such that X is (c,d)-wide over W and $K = \mathrm{comp}_G(W)$. Let $w \in V(K_0)$ be ℓ-central in W, and let $C_1, \ldots, C_\ell \subseteq W$ be the genericℓ-nest in W. Then $\rho(w) \in \mathrm{int}(\Gamma(C_1))$. We let W' be a subwall of W of height $h-1$ such that $w \notin V(W')$. (To obtain W', we delete the row and column of W that contains W; if there is no such row and/or column, we just take the row and column closest to w.) Then $(\rho, W', K_0 - w, K_1, \ldots, K_n)$ is a flat $(h-1)$-wall decomposition in $G \setminus (X \cup \{w\})$.

Let (T, Y) an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of $G - w$. We choose the node $t \in V(T)$ according to Lemma 4.3 (applied to W', so as a first condition on ℓ we need to make sure that $h - 1 > 4\alpha$). We let $H = H_t$ be the torso of (T, Y) at t and $(\sigma, H_0, Z, \mathcal{V}, \emptyset)$ a nice $(\alpha_1, \alpha_2', \alpha_3)$-near embedding of H in a surface Σ of Euler genus $\gamma' \leq \gamma$ satisfying all the conditions stated in the definition of $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decompositions in Section 3.3.

Now the idea of the proof of the Extension Lemma is to prove that one of the cycles $C = C_i$ of the ℓ-nest not only bounds the disk $\Gamma(C) \subseteq \Gamma$ under the embedding ρ of K, but also a disk $\Delta \subseteq \Sigma$ under the embedding σ of H. This will be the content of Lemma 4.4 below. Once we have proved this, we can identify the disk $\Gamma(C)$ with the disk Δ and modify the embedding σ so that it coincides with ρ on $\Delta = \Gamma(C)$ and remains unchanged on $\Sigma \setminus \Delta$. We can easily extend this new embedding to $H + w$, because ρ is defined on K and not just $K - w$. Thus we can insert w in the bag Y_t and this way extend the decomposition from $G - w$ to G. Moreover, we can do this extension algorithmically in linear time if we are given the flat wall decomposition and the embedding σ of H_0 in Σ.

A technical difficulty with this approach is that so far we ignored the set X. If vertices of X are mapped into the disk Δ by σ, then we cannot modify σ in the way described, because ρ is not defined on X. This is where we use the wideness condition on X. Suppose for contradiction that $\sigma(x) \in \Delta$ for some $x \in X$. Then if $c \geq \alpha_1 + 2$, there is a set of at least $\alpha_1 + 2$ neighbours of x in $\mathrm{comp}_{G-X}(W)$ that are mutually far apart, and at least two of these neighbours, say, y_1, y_2, are not in Z, because $|Z| \leq \alpha_1$. Now both y_1 and y_2 must be mapped into Δ as well, because $\sigma(x) \in \Delta \setminus \sigma(C) = \mathrm{int}(\Delta)$. But y_1 and y_2 are far apart with respect to W. Hence the subgraph of K_0 in the interior of the disk $\Gamma(C)$ together with x is not planar, because it contains a relatively large subwall and edges spanning several bricks of this subwall. However, σ embeds the subwall in the disk Δ. This is a contradiction.

Thus what remains to prove is that indeed there is such a cycle $C = C_i$ of the ℓ-nest that bounds a disk under σ. To prove this, it will be convenient to extend the near embedding $(\sigma, H_0, Z, \mathcal{V}, \emptyset)$ from H to G. For each vortex $(J, \Omega) \in \mathcal{V}$, we define a new vortex (J', Ω) by letting J' be the union of J with the induced subgraph $G[\bigcup_u Y_u] - Z$, where the union ranges over all nodes u of T that are contained in a component of $T - t$ attached to the vortex J. We let $\mathcal{V}'$ be the resulting set of vortices. For each child u of t attached to G_0 we define a society (J, Ω) with $J = G[\bigcup_{u'} Y_{u'}] - Z$, where the union ranges over all u' in the subtree rooted at u, and with $V(\Omega) = (Y_t \cap Y_u) \setminus Z$. Then $|V(\Omega)| \leq 3$. We let $\mathcal{W}'$ be the set of all societies defined this way. This yields an $(\alpha_1, \alpha_2, \alpha_3)$-near embedding of G in Σ. Observe that the wall W has the following two properties with respect to this near embedding:

(1) For every $(J, \Omega) \in \mathcal{V}'$ there is a linear decomposition $(X_i)_{i \in [m-1]}$ of depth α_3 such that for every $i \in [m]$ at most $16\alpha^2$ nails of W are contained in X_i.

(2) For every $(J, \Omega) \in \mathcal{W}$, at most $16\alpha^2$ nails of W are contained in J.

Condition (2) follows immediately from our choice of t according to Lemma 4.3. To see (1), let $(J', \Omega') \in \mathcal{V}'$, and let (J, Ω) be the corresponding vortex in $\mathcal{V}$. Then (J, Ω) has a linear decomposition $(X_i)_{i \in [m-1]}$ of depth at most α_3 and width at most $2\alpha_3 + 1$. We obtain a linear decomposition $(X_i')_{i \in [m-1]}$ of depth α_3 of J' by attaching to each X_i the bags of the subtrees attached to X_i. Then the set X_i separates X_i' from the rest of G. Thus by Lemma 4.2, either X_i' or $G \setminus (X_i' \setminus X_i)$ contains at most $(2\alpha_3 + 1)^2$ nails of W. By the choice of t, it must be X_i'.

If W is a wall that satisfies conditions (1) and (2) for a near embedding $(\sigma, G_0, Z, \mathcal{V}', \mathcal{W})$, we say that the near embedding is *W-central*.

Thus the Extension Lemma 4.1 follows from the following lemma. Recall the definition of the graph G_0' (for a near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$) and the shortcut C' for a cycle $C \subseteq G_0 \cup \bigcup_{(J,\Omega) \in \mathcal{W}} J$ in G_0'. We say that C bounds a disk $\Delta \subseteq \Sigma$ if $\mathrm{bd}(\Delta) = \sigma'(C')$.

LEMMA 4.4. *For all nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \gamma, \xi$ there are nonnegative integers c, d, ℓ, h such that the following holds. Let $X \subseteq V(G)$ of size $|X| \leq \xi$, and let $(\rho, W, K_0, \ldots, K_n)$ be a flat wall decomposition of height h in $G - X$ such that X is (c,d)-wide over W. Let $C_1, \ldots, C_\ell \subseteq W$ be the generic ℓ-nest in W. Let $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ be a nice $(\alpha_1, \alpha_2, \alpha_3)$-near embedding of a graph G in a surface Σ of Euler genus at most γ that is W-central. Then there is some $i \in [\ell]$ such that $C_i \subseteq G_0 \cup \bigcup_{(J,\Omega) \in \mathcal{W}} J$ and C_i bounds a disk in Σ.*

The rest of this section is devoted to a proof of Lemma 4.4. The proof is by induction on the lexicographical order of pairs (γ, α_2) (that is, the genus of the surface and the number of vortices).

We let $\alpha_1, \alpha_2, \alpha_3, \gamma, \xi$ be nonnegative integers and $\alpha = \max\{\alpha_1, \alpha_2, \alpha_2\}$. We let c, d, ℓ, h be sufficiently large (to be determined later). We choose the graph G, the set X, the flat wall decomposition $(\rho, W, K_0, \ldots, K_n)$, the ℓ-nest $C_1, \ldots, C_\ell$, and the near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ in Σ as in the statement of the lemma. We let $K = \text{comp}_{G-X}(W)$.

4.1 Base Cases If $\gamma = \alpha_2 = 0$, then Σ is the sphere and $G_0 = G \setminus Z$. We just make sure that $\ell > \alpha_1 \geq |Z|$, then one of the cycles C_i has an empty intersection with Z. Then $C_i \subseteq G_0 \cup \bigcup_{(J,\Omega) \in \mathcal{W}} J$, because $\mathcal{V} = \emptyset$, and C_i trivially bounds a disk in the sphere Σ.

The more interesting base case is $\gamma = 0$ and $\alpha_2 = 1$, that is, Σ is a sphere and there is exactly one vortex: $\mathcal{V} = \{(G_1, \Omega_1)\}$. Suppose that $V(\Omega_1) = \{x_0, \ldots, x_n - 1\}$, where $x_i = \Omega_1^i(x_0)$, and let $(X_i : 0 \leq i < n)$ be a linear decomposition of (G_1, Ω_1) of depth α_3 such that $x_i \in X_i$.

Claim 4.1. *G_1 does not contain a subwall of W of height 4α.*

Proof. Let $W' \subseteq G_1$ be a subwall of W. For every $i \in [0, m-1]$, let $Y_i^{\leq} = \bigcup_{j=0}^{i} X_j$ and $Y_i^{>} = \bigcup_{j=i+1}^{m-1} X_j$. Then $Y_i^{\leq} \cap Y_i^{>} = X_i \cap X_{i+1}$ and thus $|Y_i^{\leq} \cap Y_i^{>}| \leq \alpha_3 \leq \alpha$. Then

(1) for every row Q of W' there is no i such that both $Y_i^{\leq}$ and $Y_i^{>}$ contain more than α nails of Q.

To see this, let $Q^{\leq}$ be the set of nails of Q in $Y_i^{\leq}$ and $Q^{>}$ be the set of nails of Q in $Y_i^{>}$. Observe that if $|Q^{\leq}| \geq \alpha + 1$ and $|Q^{>}| \geq \alpha + 1$ then there are $\alpha + 1$ mutually disjoint paths from $Q^{\leq}$ to $Q^{>}$ in $W' \subseteq G_1$: just take the columns of the nails in $Q^{\leq} \cup Q^{>}$ in W' and connect them appropriately by rows of W'. But this leads to a contradiction, because the set $X_i \cap X_{i+1}$ of size at most α separates $Q^{\leq} \subseteq Y_i^{\leq}$ from $Q^{>} \subseteq Y_i^{>}$. This proves (1).

It follows from (1) that for every row Q of W' there is an $i = i(Q)$ such that $X_{i(Q)}$ contains all but at most $2\alpha + 1$ nails of row Q. Furthermore, for rows Q, Q' we have $i(Q) = i(Q')$, because there are more than α disjoint paths the nails of Q in $X_{i(Q)}$ and the nails of Q' in $X_{i(Q')}$. Hence if the height of W' is h', there is an $i = i(Q)$ such that X_i contains all but $(h'+1)(2\alpha+1)$ nails of W'. Overall, W' contains $2(h'-1)(h'+1)$ nails. Thus X_i contains at least $(2h' + 2\alpha - 1)(h' + 1)$ nails. As our near embedding is W-central, it follows that $(2h' + 2\alpha - 1)(h' + 1) \leq 16\alpha^2$, which implies $h' < 4\alpha$ □

Now we are ready to prove the key claim of this section, and in a sense of the whole proof of the Extension Lemma. Recall that $K = \text{comp}_{G-X}(W)$.

Claim 4.2. *Suppose that the height h of W and ℓ are sufficiently large (in terms of α and ξ). Then every vertex of K that is ℓ-central in W is contained in $V(G_0) \cup \bigcup_{(J,\Omega) \in \mathcal{W}} V(J)$.*

For the ease of presentation in this conference version, we make the simplifying assumption that $Z = \emptyset$.

Proof. Suppose for contradiction that the claim is false. Consider a counterexample where $|G|$ is minimal, and among all such counterexamples, choose one where $|\mathcal{W}|$ is minimal.

Let $\Delta = \Sigma \setminus \text{int}(\Delta_1)$, where Δ_1 is the disk that accommodates the vortex (G_1, Ω_1). Then Δ is a closed disk and σ an embedding of G_0 in Δ.

We first prove that $\mathcal{W} = \emptyset$. Suppose for contradiction that $(J, \Omega) \in \mathcal{W}$. Let $\Delta_J \subseteq \Delta$ be the disk that accommodates (J, Ω). If $|V(J) \setminus V(\Omega)| \leq 1$, we can extend the embedding σ to $G_0 \cup J$, and by letting $G_0^- = G_0 \cup J$ and $\mathcal{W}^- = \mathcal{W} \setminus \{(J, \Omega)\}$ we obtain a counterexample with the same G and smaller $\mathcal{W}$, which contradicts the minimality. So $|V(J) \setminus V(\Omega)| \geq 2$. Now $V(J) \setminus V(\Omega)$ contains at most one nail of W, because $Z = \emptyset$ and therefore $|V(G_0) \cap V(J)| \leq 3$. We can replace J by a smaller graph J^* that in addition to the vertices in $V(\Omega)$ has just one vertex v^* which is connected to all vertices in $V(\Omega)$. If there is a nail of W in J, we can replace this nail by v^*. This way we obtain a smaller counterexample to the claim. This proves that $\mathcal{W} = \emptyset$ and thus $G = G_0 \cup G_1$.

By an argument given above, each vertex in X is contained in G_1, because otherwise it would destroy the planarity of G_0. We may furtgher assume assume that $G - X$ is not planar, because if ℓ is sufficiently large we cannot have ℓ nested cycle in a planar vortex of depth α_3.

Let us consider a graph G' that is obtained from G by deleting $V(K)$ and X. Then G' is not planar with the outer face boundary *per*(W) (i.e, the perimeter of W), because otherwise $G - X$ would be planar as well.

(1) G_1 contains a vertex in $G - X - V(K)$.

Recall that our near embedding is nice, and let $C' \subseteq G_0$ be a cycle with $\text{bd}(\Delta_1) = \sigma(C')$, where Δ_1 is the disk that accomodates the vortex (G_1, Ω_1).

Let us first consider the case when $X = \emptyset$. Suppose that there is a vertex v in $G_1 \cap K$ that is at least $4\alpha + 1$-central in W. Since $X = \emptyset$, by Claim 4.1, this implies that

(2) There is a path P that is a subpath of C' such that P is contained in K, the two endvertices of P are on $per(W)$, and moreover, the planar graph $Q \subseteq G_1$ containing v that is bounded by P together with a segment $per(W)$ does not contain a subwall of W of height 4α.

By (2) and since v is $4\alpha+1$-central, P contains a vertex u that is $2\alpha+1$-central. Let P', P'' be $P-u$. Then both P' and P'' hit the same $2\alpha+1$ rows of W, $R_0, R_1, \ldots, R_{2\alpha}$, and hence there are $2\alpha+1$ disjoint paths from P' and P''. Since both P' and P'' are segments of C', this contradicts G_1 being a vertex of depth $\alpha_3 \leq \alpha$.

Finally, assume $X \neq \emptyset$. By the same argument as above, it follows that a path P as in (2) does not exist.

We claim the following.

(3) Let $x \in X$. Then there is no connected subgraph $R \subseteq G_1$ containing two neighbors x_1, x_2 of x that are of face-distance in W at least $(2\xi+8)\alpha^2$ such that R contains a path P connecting x_1 and x_2 in K.

Suppose such a connected subgraph R exists. We may assume that R has a nonempty intersection with at least $(2\xi+8)\alpha^2$ rows of W. If at least $8\alpha+1$ rows of W can go to $per(W)$ from R without hitting any vertex in C', then $4\alpha+1$ of these paths would hit at least $4\alpha+1$ columns in G_1, a contradiction to Claim 4.1.

Thus at least $2\xi\alpha$ paths cannot go to $per(W)$ from R without hitting any vertex in C'. Let $C'' = C' - X$. Since $|X| \leq \xi$, thus C'' consists of at most $l \leq \xi$ paths $P_1, \ldots, P_l$. Since K is planar, if a subpath of P_j in K hits two columns L_1, L_2, it must hit all the columns between L_1 and L_2 in W. This implies that either there is a path P_i and a vertex u' of P_i such that both components of $P_i - u'$ hit the same $2\alpha+1$ columns or rows (for some i), or there are two paths $P_j, P_{j'}$ such that both P_j and P'_j hit the same $2\alpha+1$ columns or rows. In both cases, G_1 cannot be a vertex of depth α, a contradiction.

Now if each vertex in X is $(\xi+1, (2\xi+8)\alpha^2)$-wide (that is, if $c \geq \xi+1$ and $d \geq 2\xi+8)\alpha^2$), (3) implies that there is a subpath P of C' satisfying (2), which is also a contradiction. This completes the proof of Claim 4.2 □

Observe that Claim 4.2 implies the base case $\gamma = 0$, $\alpha_2 = 1$ of Lemma 4.4, because for sufficiently large ℓ it implies that the innermost cycle C_1 of the ℓ-nest is in G_0 and hence bounds a disk in the sphere.

4.2 Inductive step Suppose that $\gamma \geq 1$ or $\gamma = 0$ and $\alpha_2 \geq 2$, and suppose that the assertion of Lemma 4.4 holds for all (γ', α'_2) lexicographically smaller than (γ, α_2).

Arguing similarly as in the proof of Claim 4.2, we may assume that $\mathcal{W} = \emptyset$. Suppose that the vortices in $\mathcal{V}$ are (G_i, Ω_i) for $i \in [\alpha_2]$. The cycle C_i of the ℓ-nest divides the graph $G-X$ into two parts Y_i and $G-X-Y_i$ such that Y_i contains all the cycles $C_1, \ldots, C_{i-1}$. We may assume that G and G_0 are connected. Assuming that ℓ is even, we let $H = G - Y_{\ell/2}$ and $H_0 = G_0 \setminus Y_{\ell/2}$. If ℓ is sufficiently large, may assume that H and H_0 are connected.

If $\alpha_2 \geq 2$, then we can find a path $P \subseteq H$ whose with endvertices in two different vortices G_i, G_j and all internal vertices in H_0. We can delete the path P and merge the two vortices into one vortex. Then we apply the inductive hypothesis to the resulting near embedding and the subwall of W with perimeter $C_{\ell/2+1}$.

If $\gamma \geq 1$ and $\alpha_2 \leq 1$, we either find a noncontractible cycle in H or a path with both endpoints in the vortex (as $\alpha_2 \leq 1$, we have at most one vortex) that can be completed to a noncontractible simple closed curve by connecting its two endpoints by an arc through the disk accommodating the vortex. If we find a noncontractible cycle, we delete it and apply the induction hypothesis. If we find a path we delete it and split the vortex in two. This gives us a near embedding of smaller genus and with one more vortex. Again we can apply the induction hypothesis.

This completes our proof (sketch) of Lemma 4.4 and the Extension Lemma 4.1. □

5 Defining the local structure in MSO

Monadic second-order logic MSO is the extension of first-order predicate logic that admits quantification not only over the individual elements of a structure, but also over sets of elements. We only introduce a specific version of MSO for graphs; in the literature this version is known as MSO_2 or GSO. Our logic uses four types of variables: *individual variables ranging* over vertices and edges, respectively, and *set variables* ranging over sets of vertices and sets of edges, respectively. *Atomic formulas* are of the form $x = x'$, where x and x' are either both vertex variables or both edge variables, $I(x, y)$, where x is a vertex variables and y an edge variable, $X(x)$, where either x is a vertex variable and X a vertex-set variable or x is an edge variable and X an edge-set variable. The formula $x = x'$ expresses equality, the formula $I(x, y)$ incidence, and the formula $X(x)$ set membership ("the element x is contained in the set X"). MSO-formulas are built from atomic formulas using the usual Boolean connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\neg$ (negation) and both existential and universal quantification over all four types of variables.

We write $\phi(X_1,\ldots,X_k)$ to indicate that the free variables (that is, the variables that have an occurrence not bound by a quantifier) are among $X_1,\ldots,X_k$. For a graph G and interpretations $X_1^G,\ldots,X_k^G$ of appropriate types for the variables, we write $G \models \phi(X_1^G,\ldots,X_k^G)$ to denote that G satisfies ϕ if X_i is interpreted by X_i^G.

Example 5.1. The following formula $\mathsf{cycle}(Y)$ with a free edge-set variable Y says that Y is the edge set of a cycle.

$$\begin{aligned}
&\mathsf{cycle}(Y) := \\
&\quad \forall x \forall y\Big(\big(Y(y) \wedge I(x,y)\big) \to \\
&\qquad \exists y'\Big(Y(y') \wedge y' \neq y \wedge I(x,y') \\
&\qquad\qquad \wedge \neg\exists y''\big(Y(y'') \wedge y'' \neq y \wedge y'' \neq y' \wedge I(x,y'')\big)\Big)\Big) \\
&\quad \wedge \forall X\Big(\forall x\big(X(x) \to \exists y(Y(y) \wedge I(x,y))\big) \\
&\qquad\qquad \wedge \exists x X(x) \wedge \exists x \exists y\big(Y(y) \wedge I(x,y) \wedge \neg X(x)\big) \to \\
&\quad \exists y \exists x \exists x'\big(Y(y) \wedge I(x,y) \wedge I(x',y) \wedge X(x) \wedge \neg X(x')\big)\Big).
\end{aligned}$$

The subformula in the first three lines says that the subgraph with edge set Y is 2-regular, and the subformula in the last three lines says that the subgraph with edge set Y is connected.

We can use the formula $\mathsf{cycle}(Y)$ to express that a graph is Hamiltonian: we let

$$\mathsf{hamiltonian} := \exists Y(\mathsf{cycle}(Y) \wedge \forall x \exists y(Y(y) \wedge I(x,y))).$$

To exploit MSO-definability algorithmically, we need the following theorem, which is a slight extension of a well-known result due to Courcelle [4] from decision to evaluation problems [2] (also see [13]).

Theorem 5.1. ([4, 2]) *For all nonnegative integers ℓ, w there is a linear time algorithm that, given a graph G of tree width at most w and an* MSO*-formula $\phi(X_1,\ldots,X_k)$ of length at most ℓ, decides if there are interpretations $X_1^G,\ldots,X_k^G$ for the variables X_i in G such that $G \models \phi(X_1^G,\ldots,X_k^G)$ and computes such interpretations if they exist.*

Our goal in this section will be to construct an MSO-formula defining near embeddings that are $\mathfrak{T}_U$-central for some tangle defined by an unbreakable set U. But what does it mean to "define" a near embedding in MSO? For this, we need to find a way to represent near embeddings in a format that is accessible in MSO, that is, as tuple of vertices, edges, vertex sets, and edge sets. The main difficulty here is to find such a format and not so much to write down the actual MSO-definitions, which is merely a tedious exercise (much like coding in assembler). The main technical result of this section is the Definability Lemma 5.7.

5.1 Vortices We say that a cyclic permutation Ω of a set $V(\Omega)$ is *compatible* with a cycle C if $V(\Omega) = V(C)$ and $v\Omega(v) \in E(C)$ for all $v \in V(C)$. A *nice α-vortex* in a graph G is a pair (H, C) such that

(1) $H \subseteq G$;

(2) $C \subseteq G$ is a cycle in G with $V(C) \subseteq V(H)$ and $E(C) \cap E(H) = \emptyset$;

(3) there is a cyclic permutation Ω of $V(C)$ that is compatible with C such that (H, Ω) is a α-vortex.

We say that a triple (X^G, Y^G, Z^G), where $X^G \subseteq V(G)$ and $Y^G, Z^G \subseteq E(G)$, *represents* a nice α-vortex (H, C) in G if $V(H) = X^G$, $E(H) = Y^G$, and $E(C) = Z^G$. Clearly, if (X^G, Y^G, Z^G) represents (H, C) then we can reconstruct (H, C) from (X^G, Y^G, Z^G), because $V(C)$ is determined by $E(C)$.

For an edge set $F \subseteq E(G)$ in a graph G, it will be convenient to denote the set of all vertices incident with an edge in F by $V(F)$.

Lemma 5.1. *For every nonnegative integer α there is an* MSO*-formula $\mathsf{vortex}_\alpha(X, Y, Z)$ with a free vertex-set variable X and free edge-set variables Y, Z such that for all graphs G and sets $X^G \subseteq V(G)$, $Y^G, Z^G \subseteq E(G)$ we have*

$$G \models \mathsf{vortex}_\alpha(X^G, Y^G, Z^G)$$

if and only if (X^G, Y^G, Z^G) represents a nice α-vortex in G.

Proof. It is easy to construct MSO-formulas $\mathsf{society}(X, Y, Z)$, $\mathsf{path}(X, Y, Z, x, y)$, $\mathsf{sep}(Z, x, y, z_1, z_2)$, and $\mathsf{disjoint}(Z_1, \ldots, Z_{\alpha+1})$ such that for all graphs G, all $X^G \subseteq V(G)$, and all $Y^G, Z^G, Z_1^G, \ldots, Z_{\alpha+1}^G \subseteq E(G)$, and all $u, v, w_1, w_2 \in V(G)$,

- $G \models \mathsf{society}(X^G, Y^G, Z^G)$ if and only if $H := (X^G, Y^G)$ is a subgraph of G (that is, $V(Y^G) \subseteq X^G$) and Z^G is the edge set of a cycle C such that $V(C) \subseteq V(H)$ and $E(C) \cap E(H) = \emptyset$.
- If $H := (X^G, Y^G)$ is a subgraph of G, then $G \models \mathsf{path}(X^G, Y^G, Z^G, u, v)$ if and only if Z^G is the edge set of a path in H with endvertices u and v.
- If Z^G is the edge set of a cycle C, then $G \models \mathsf{sep}(Z^G, u, v, w_1, w_2)$ if and only if u, v, w_1, w_2 are mutually distinct vertices of C and $\{w_1, w_2\}$ separates u from v in C.

- $G \models \mathsf{disjoint}(Z_1^G, \dots, Z_{\alpha+1}^G)$ if and only if for $1 \leq i < j \leq \alpha+1$ we have $V(Z_i^G) \cap V(Z_j^G) = \emptyset$.

Then we let

$$\begin{aligned}\mathsf{vortex}_\alpha(X,Y,Z) := \mathsf{society}(X,Y,Z) \wedge \\ \neg\exists Z_1 \dots \exists Z_{\alpha+1} \exists x_1 \dots \exists x_{\alpha+1} \exists y_1 \dots \exists y_{\alpha+1} \exists z_1 \exists z_2 \\ \Big(\mathsf{disjoint}(Z_1, \dots, Z_{\alpha+1}) \\ \wedge \bigwedge_{i=1}^{\alpha+1} \big(\mathsf{path}(X,Y,Z_i,x_i,y_i) \\ \wedge \mathsf{sep}(Z, x_i, y_i, z_1, z_2)\big)\Big).\end{aligned}$$

This formula says that Z is the edge set of cycle with vertices in the subgraph (X,Y) and edge set disjoint from the subgraph and that there is no separator $\{z_1, z_2\}$ of the cycle Z and disjoint paths $Z_1, \dots, Z_{\alpha+1}$ in (X,Y) from one side of the separation to the other side. This expresses precisely that (X,Y,Z) represents a nice α-vortex. □

5.2 Near Embeddings

LEMMA 5.2. ([1]) *For every nonnegative integer γ there is an MSO-sentence emb_γ satisfied by a graph G if and only if G is embeddable in a surface of Euler genus at most γ.*

Proof sketch. The proof is by induction on γ. The sentence emb_0 just says that neither K_5 nor $K_{3,3}$ is a minor. The sentence $\mathsf{emb}_{\gamma+1}$ guesses the edge set of a noncontractible cycle, and a partition of the edges incident to a vertex of the cycle that says which edges go on which side of the cycle (one part of the partition may be empty). From the cycle and the partition, we can construct the graph obtained by cutting along the cycle, and we say that this graph satisfies $\mathsf{emb}_{\gamma-1}$ (in the orientable case) or emb_γ (in the nonorientable case). □

REMARK 5.1. *Note that the proof of the lemma just sketched not only gives us a sentence emb_γ, but also an algorithm that, given γ computes such a sentence.*

If we were only interested in the existence of a sentence, we could also use the fact that the class of all graphs embeddable in a surface of Euler genus at most γ is characterised by finitely many forbidden minors and let emb_γ just state that these input graphs contains non of these forbidden minors. (This approach can also be made effective by using a result due to Seymour [31], which says that there is a computable function that associates with each γ a set of forbidden minors for the corresponding class.)

LEMMA 5.3. *For all nonnegative integers α, γ there is an MSO-formula $\mathsf{disks}_{\alpha,\gamma}(Z_1 \dots, Z_\gamma)$ with a free edge-set variables $Z_1, \dots, Z_\alpha$ such that for all graphs G and all $Z_1^G, \dots, Z_\alpha^G \subseteq E(G)$ we have*

$$G \models \mathsf{disks}_{\alpha,\gamma}(Z_1, \dots, Z_\alpha)$$

if and only if the following conditions are satisfied.

(1) $Z_1^G, \dots, Z_\alpha^G$ are the edge sets of mutually disjoint cycles $C_1, \dots, C_\alpha \subseteq G$.

(2) There is an embedding σ of G in a surface Σ of Euler genus at most γ and mutually disjoint closed disks $\Delta_1, \dots, \Delta_\alpha \subseteq \Sigma$ such that $\sigma(G) \cap \mathrm{int}(\Delta_i) = \emptyset$ and $\sigma(C_i) = \mathrm{bd}(\Delta_i)$ for all $i \in [\alpha]$.

Proof. There are different ways of proving this lemma; all require some background on graph embeddings. The one we choose is probably not the simplest, but it nicely builds on the techniques we already used in Section 4. It follows from Lemma 4.4 that for every γ there is an h such that for all graphs G embedded in a surface of Euler genus γ, if W is a wall of height h in G such that $\mathrm{comp}_G(W)$ is planar, then (each of) the central brick(s) of W must bound a disk.

We exploit this as follows. Suppose we have a graph G and cycles $C_1, \dots, C_\alpha \subseteq G$, and we want to test if G has an embedding in a surface of Euler genus at most γ where the cycles bound disks that are faces of the embedding. We proceed as follows: for each cycle C_i we add a wall W_i of height h. We draw disjoint edges from the vertices of C_i to the perimeter of W_i, preserving the cyclic order. If there are not enough vertices on the perimeter, we subdivide some of the edges. Let G' be the resulting graph. We claim that for every surface Σ of Euler genus at most γ, the graph G has an embedding where the cycles C_i bound disks Δ_i that are faces of the embedding if and only if G' has an embedding in Σ. The forward direction of this claim is obvious. To prove the backward direction, consider an embedding σ' of G' in Σ. For every i, take a disk Δ_i' bounded by a central brick of W_i. We modify the embedding by drawing the whole wall W_i in the disk Δ_i, with the perimeter on the boundary of the disk, and routing the edges from C_i to the perimeter in close neighbourhoods of the σ-images of paths from the perimeter to the central brick in W_i. Let σ' be the resulting embedding. Now we let Δ_i be the union of Δ_i' with small neighbourhoods of the σ'-images all the edges from C_i to the perimeter of W_i. (So the disk Δ_i has the shape of an octopus with a body in the disk Δ_i' and arms reaching out to the vertices of C_i.) We can redraw the cycle C_i by keeping the vertices where they are and drawing the edges along the boundary of the disk Δ_i. This proves the claim. Thus to test if G

has an embedding of the desired form, we can test if G' has an embedding in a surface of Euler genus γ.

To formalise this test in MSO, we define the graph G' within the graph G, given the (edge sets of the) cycles $C_1, \ldots, C_\alpha$. We can do this by an MSO-*transduction* (see, e.g., [5]). This is straightforward, though tedious. Once we have defined G', we can use Lemma 5.2 to test (in MSO) if it can be embedded in a surface of Euler genus at most γ. □

Corollary 5.1. *For all nonnegative integers α, γ there is an MSO-formula $\mathsf{nice\text{-}emb}_{\alpha,\gamma}(X, Y, Z_1, \ldots, Z_\alpha)$ with a free vertex-set variable X and free edge-set variables $Y, Z_1, \ldots, Z_\alpha$ such that for all graphs G, all $X^G \subseteq V(G)$ and $Y^G, Z_1^G, \ldots, Z_\alpha^G \subseteq E(G)$ we have*

$$G \models \mathsf{nice\text{-}emb}_{\alpha,\gamma}(X^G, Y^G, Z_1^G, \ldots, Z_\alpha^G)$$

if and only if the following conditions are satisfied.

(1) $H := (X^G, Y^G)$ is a subgraph of G.

(2) For all $i \in [\alpha]$ the set Z_i^G is the edge set of a cycle $C_i \subseteq H'$, where H' is the graph obtained from H by adding an edge between any two nonadjacent vertices v, w such that there is a connected components A of $G \setminus V(H)$ with $v, w \in N^G(A)$.

(3) The cycles $C_1, \ldots, C_\alpha$ are mutually disjoint.

(4) There is an embedding σ of H' in a surface Σ of Euler genus at most γ.

(5) There are closed disks $\Delta_1, \ldots, \Delta_n \subseteq \Sigma$, for some $n \geq \alpha$, with disjoint interiors such that for all $i \in [n]$ we have $\sigma(H) \cap \operatorname{int}(\Delta_i) = \emptyset$.

(6) For $1 \leq i \leq \alpha$, the simple closed curve $\sigma(C_i)$ is the boundary of the disk Δ_i.

(7) For $\alpha + 1 \leq i \leq n$ there is a connected component A of $G \setminus V(H)$ such that $\sigma(V(G_0)) \cap \operatorname{bd}(\Delta_i) = \sigma(N^G(A))$.

(8) For every connected component A of $G \setminus V(H)$ there is an i, $\alpha + 1 \leq i \leq n$, such that $\sigma(N^G(A)) \subseteq \operatorname{bd}(\Delta_i)$.

In the following lemmas, $\bar{z} = (z_1, \ldots, z_{\alpha_1})$ denotes an α_1-tuple of vertex variables, $\bar{X} = (X_0, \ldots, X_{\alpha_2})$ an $(\alpha_2 + 1)$-tuple of vertex-set variables, $\bar{Y} = (Y_0, \ldots, Y_{\alpha_2})$ an $(\alpha_2 + 1)$-tuple of edge-set variables, and $\bar{Z} = (Z_1, \ldots, Z_{\alpha_2})$ an α_2-tuple of edge-set variables. We denote tuples of interpretations for these variables in a graph G by $\bar{z}^G = (z_1^G, \ldots, z_{\alpha_1}^G)$, $\bar{X}^G = (X_0^G, \ldots, X_{\alpha_2}^G)$ et cetera.

Lemma 5.4. *For all nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \gamma$ there is an MSO-formula*

$$\mathsf{near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\gamma}(\bar{z}, \bar{X}, \bar{Y}, \bar{Z})$$

such that for all graphs G and all $\bar{z}^G \in V(G)^{\alpha_1}$, $\bar{X}^G \in (2^{V(G)})^{\alpha_2+1}$, $\bar{Y}^G \in (2^{E(G)})^{\alpha_2+1}$, $\bar{Z}^G \in (2^{E(G)})^{\alpha_2}$,

$$G \models \mathsf{near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\gamma}(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$$

if and only if the following conditions are satisfied.

(1) For $1 \leq i \leq \alpha_1$ and $0 \leq j \leq \alpha_2$, we have $z_i^G \notin X_j^G$. Let $Z := \{z_1^G, \ldots, z_{\alpha_1}^G\}$.

(2) For $i = 0, \ldots, \alpha_2$, the pair $G_i := (X_i^G, Y_i^G)$ is a subgraph of G. Let $H := \bigcup_{i=0}^{\alpha_2} G_i$.

(3) For $1 \leq i \leq \alpha_2$, the set Z_i^G is the edge set of a cycle $C_i \subseteq G_0'$, where G_0' is the graph obtained from G_0 by adding an edge between any two nonadjacent vertices $v, w \in V(G_0)$ such that there is a connected components A of $G_0 \setminus (V(H) \cup Z)$ with $v, w \in N^G(A)$.

(4) For $1 \leq i \leq \alpha_2$, we have $V(G_i) \cap V(G_0) = V(C_i)$ and $E(G_i) \cap E(G_0) = \emptyset$.

(5) For $1 \leq i < j \leq \alpha_2$, we have $G_i \cap G_j = \emptyset$.

(6) For $1 \leq i \leq \alpha_2$ the pair (G_i, C_i) is a nice α_3-vortex.

(7) For all connected components A of $G \setminus (V(H) \cup Z)$, let $N_A := N^G(A) \setminus Z$. Then $N_A \subseteq V(G_0)$ and $|N_A| \leq 3$.

(8) There is an embedding σ of G_0' in a surface Σ of Euler genus at most γ.

(9) There are mutually disjoint closed disks $\Delta_1, \ldots, \Delta_n \subseteq \Sigma$, for some $n \geq \alpha_2$, such that for all $i \in [n]$ we have $\sigma(G_0) \cap \operatorname{int}(\Delta_i) = \emptyset$.

(10) For $i = 1, \ldots, \alpha_2$, the simple closed curve $\sigma(C_i)$ is the boundary of the disk Δ_i.

(11) For $i = \alpha_2 + 1, \ldots, n$ there is a connected components A of $G \setminus (V(H) \cup Z)$ such that $\sigma(V(G_0)) \cap \operatorname{bd}(\Delta_i) = \sigma(N_A)$.

(12) For all connected components A of $G \setminus (V(H) \cup Z)$ there is an i, $\alpha_2 + 1 \leq i \leq n$, such that $\sigma(N_A) \subseteq \operatorname{bd}(\Delta_i)$.

Proof. This follows easily from Lemma 5.1 and Corollary 5.1. □

Let G be a graph. We say that a tuple $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G) \in V(G)^{\alpha_1} \times (2^{V(G)})^{\alpha_2+1} \times (2^{E(G)})^{\alpha_2+1} \times (2^{E(G)})^{\alpha_2}$ *represents* a nice $(\alpha_1, \alpha_2, \alpha_3)$-near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ of G in a surface Σ if

$$G \models \mathsf{near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\gamma}(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$$

and $Z = \{z_1^G, \dots, z_{\alpha_1}^G\}$ and $G_0 = (X_0^G, Y_0^G)$, and with G_i, C_i defined as in (2), (3) we have $\mathcal{V} = \{(G_i, \Omega_i) \mid 1 \le i \le \alpha_2\}$, where Ω_i is a cyclic permutation of $V(C_i)$ compatible with C_i, and (8)–(12) hold for the surface Σ and the embedding σ.

For every tuple $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ with $G \models \mathsf{near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\gamma}(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ there is a nice $(\alpha_1, \alpha_2, \alpha_3)$-near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ of G in a surface Σ represented by $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$. Whereas $\sigma, G_0, Z, \mathcal{V}$ are immediately given by the tuple $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ and (1)–(12), we need to define $\mathcal{W}$. We let $A_1, \dots, A_k$ be the connected components of $G \setminus (V(H) \cup Z)$. We inductively define sets $\mathcal{W}_0, \mathcal{W}_1, \dots, \mathcal{W}_k$ of societies. We let $\mathcal{W}_0 = \emptyset$. To define $\mathcal{W}_{i+1}$, we look at the component $A = A_{i+1}$. If there is a society $(J, \Omega) \in \mathcal{W}_i$ such that $N_A = N^G(A) \setminus Z \subseteq V(\Omega)$, we let $J' = J \cup G[V(A) \cup N_A]$ and $\mathcal{W}_{i+1} := (\mathcal{W}_i \setminus \{(J, \Omega)\}) \cup \{(J', \Omega)\}$. Otherwise, we choose an arbitrary cyclic permutation Ω of N_A and let $\mathcal{W}_{i+1} := \mathcal{W}_i \cup \{(G[V(A) \cup N_A], \Omega)\}$. We let $\mathcal{W} = \mathcal{W}_k$.

Conversely, it is not hard to see that for every nice near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ with $Z \neq \emptyset$ there is a tuple $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ representing it. We may restrict our attention to near embeddings with nonempty Z, because as long as $\alpha_1 > 0$, we can always add an arbitrary element to Z if it is empty.

Thus the formula $\mathsf{near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\gamma}$ essentially defines near embeddings. Our final task is to define only such near embeddings that are $\mathfrak{T}$-central, for a tangle $\mathfrak{T} = \mathfrak{T}_U$ coming from an unbreakable set. For the following lemma, recall Lemma 3.1.

LEMMA 5.5. *Let $\mathfrak{T}$ be a tangle of order $\beta > \alpha_1 + 3$ in a graph G. Let $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ be a tuple that represents an $(\alpha_1, \alpha_2, \alpha_3)$-near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ of G in a surface Σ, and let $Z, G_0, \dots, G_{\alpha_2}, H$ be defined as above. Then $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ is $\mathfrak{T}$-central if and only if for all $S \subseteq V(G)$ with $|S| < \beta$ and $Z \subseteq S$ the following two conditions are satisfied. Let $C = C_{\mathfrak{T},S}$ and let $\hat{C}$ be the graph with vertex set $V(\hat{C}) = V(C) \cup S$ and edge set $E(\hat{C}) = E(C) \cup \{vw \in E(G) \mid v \in V(C), w \in S\}$.*

(1) There is no $i \le \alpha_2$ such that $\hat{C} \setminus Z \subseteq G_i$.

(2) There is no connected component A of $G \setminus (Z \cup V(H))$ such that $\hat{C} \subseteq \hat{A}$, where $\hat{A}$ is the graph with $V(\hat{A}) = V(A) \cup N^G(A) \cup Z$ and $E(\hat{A}) = E(A) \cup \{vw \in E(G) \mid v \in V(A), w \in N^G(A) \cup Z\}$.

In view of Lemma 3.1, this lemma essentially says that the "big" part of a separation (with respect to $\mathfrak{T}$ in must not be contained in an element of $\mathcal{V} \cup \mathcal{W}$ and thus rephrases the definition of the near embedding being $\mathfrak{T}$-central. We omit the proof, which is surprisingly tedious, though essentially just a straightforward application of the definitions, the tangle axioms, and Lemma 3.1.

COROLLARY 5.2. *Let $\mathfrak{T}$ be a tangle of order $> \alpha_1 + 3$ in a graph G. Let $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ be a tuple that represents a nice $(\alpha_1, \alpha_2, \alpha_3)$-near embedding of G that is $\mathfrak{T}$-central. Then all nice $(\alpha_1, \alpha_2, \alpha_3)$-near embeddings of G represented by $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ are $\mathfrak{T}$-central.*

In the following two lemmas, $\bar{x} = (x_1, \dots, x_{3\beta-2})$ denotes a $(3\beta-2)$-tuple of vertex variables and $\bar{x}^G = (x_1^G, \dots, x_{3\beta-2}^G)$ its interpretation in a graph G.

LEMMA 5.6. *For every positive integer β there is an* MSO*-sentence* $\mathsf{unbreakable}_\beta(\bar{x})$ *such that for every graph G and $\bar{x}^G \in V(G)^{3\beta-2}$,*

$$G \models \mathsf{unbreakable}_\beta(\bar{x}^G)$$

if and only if the vertices $x_1^G, \dots, x_{3\beta-2}^G$ are pairwise distinct, and the set $\{x_1^G, \dots, x_{3\beta-2}^G\}$ is β-unbreakable in G.

Proof. For all nonnegative integers k, ℓ, m, we let $\mathsf{sep}_{k,\ell,m}(x_1, \dots, x_k, y_1, \dots, y_\ell, z_1, \dots, z_m)$ be an MSO-formula stating that $\{z_1, \dots, z_m\}$ separates $\{x_1, \dots, x_k\}$ from $\{y_1, \dots, y_\ell\}$. Using these formulas, for all partitions (I, J, K) of $[3\beta-2]$ and all nonnegative integers m we obtain a formula $\mathsf{sep}_{(I,J,K),m}(x_1, \dots, x_{3\beta-2}, y_1, \dots, y_m)$ saying that $\{x_k \mid k \in K\} \cup \{y_1, \dots, y_m\}$ separates $\{x_i \mid i \in I\}$ from $\{x_j \mid j \in J\}$. We let

$$\mathsf{unbreakable}_\beta(x_1, \dots, x_{3\beta-2}) := \bigwedge_{1 \le i < j \le 3\beta-2} x_i \neq x_j$$
$$\wedge \neg \bigvee_{(I,J,K),m} \exists y_1 \dots \exists y_m \Big(\bigwedge_{i=1}^{3\beta-2} \bigwedge_{\ell=1}^{m} x_i \neq y_\ell$$
$$\wedge\, \mathsf{sep}_{(I,J,K),m}(x_1, \dots, x_{3\beta-2}, y_1, \dots, y_m) \Big),$$

where the disjunction ranges over all partitions (I, J, K) of $[3\beta-2]$ and all nonnegative integers m such that $|K|+m < \beta$ and $|I|+|K|+m < 3\beta-2$ and $|J|+|K|+m < 3\beta-2$. □

LEMMA 5.7. (DEFINABILITY LEMMA) *For all nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ where $\beta > \alpha_2 + 3$ there is an* MSO*-formula*

$$\mathsf{tcen\text{-}near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\beta,\gamma}(\bar{x}, \bar{z}, \bar{X}, \bar{Y}, \bar{Z})$$

such that for all graphs G, all $\bar{x}^G \in V(G)^{3\beta-2}$ and $\bar{z}^G \in V(G)^{\alpha_1}$ and $\bar{X}^G \in (2^{V(G)})^{\alpha_2+1}$ and $\bar{Y}^G \in (2^{E(G)})^{\alpha_2+1}$ and $\bar{Z}^G \in (2^{E(G)})^{\alpha_2}$,

$$G \models \mathsf{tcen\text{-}near\text{-}emb}_{\alpha_1,\alpha_2,\alpha_3,\gamma}(\bar{x}^G, \bar{z}^G \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$$

if and only if the following conditions are satisfied.

(1) The vertices $x_1^G, \dots, x_{3\beta-2}^G$ are pairwise distinct, and the set $U = \{x_1^G, \dots, x_{3\beta-2}^G\}$ is β-unbreakable in G.

(2) The tuple $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ represents a $\mathfrak{T}_U$-central nice $(\alpha_1, \alpha_2, \alpha_3)$-near embedding of G in a surface of Euler genus at most γ.

Proof. It is easy to see that there is a formula

$$\mathsf{tanglecomp}(x_1, \dots, x_{3\beta-2}, y_1, \dots, y_{\beta-1}, z)$$

such that such that for all graphs G and all $x_1^G, \dots, x_{3\beta-2}^G, y_1^G, \dots, y_{\beta-1}^G, z^G \in V(G)$, if $x_1^G, \dots, x_{3\beta-2}^G$ are pairwise distinct, and the set $U = \{x_1^G, \dots, x_{3\beta-2}^G\}$ is β-unbreakable in G, then

$$G \models \mathsf{tanglecomp}(x_1^G, \dots, x_{3\beta-2}^G, y_1^G, \dots, y_{\beta-1}^G, z^G)$$

if and only if z^G is a vertex of the component $C_{\mathfrak{T},S}$ for the tangle $\mathfrak{T} = \mathfrak{T}_U$ and the set $S = \{y_1^G, \dots, y_{\beta-1}^G\}$. The formula $\mathsf{tanglecomp}$ just has to say that for the component C of $G \setminus S$ that contains z^G it holds that $|(V(C) \cap U) \cup S| \geq 3\beta - 2$.

Using the formula $\mathsf{tanglecomp}$, we can define the conditions of Lemma 5.5 in MSO, and the lemma follows from Lemmas 5.4, 5.5 and 5.6. □

For the following corollary, recall the definition of locally $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposable graphs from Section 3.3.

Corollary 5.3. *For all nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ such that $\beta > \alpha_1 + 3$ there is an MSO-sentence $\mathsf{loc\text{-}dec}_{\alpha_1,\alpha_2,\alpha_3,\beta,\gamma}$ such that for all graphs G,*

$$G \models \mathsf{loc\text{-}dec}_{\alpha_1,\alpha_2,\alpha_3,\beta,\gamma}$$

if and only if G is locally $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposable.

5.3 Algorithmic Application

Theorem 5.2. *Let $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ be nonnegative integers such that $\beta > \alpha_1 + 3$. Then there is a linear time algorithm that, given a graph G and a β-unbreakable set $U \subseteq V(G)$ of size $|U| = 3\beta - 2$, computes a $\mathfrak{T}_U$-central nice $(\alpha_1, \alpha_2, \alpha_3)$-near embedding of G in a surface of Euler genus at most γ if such a near embedding exists.*

Proof. Using Lemma 5.7 and Theorem 5.1, we can compute a tuple $(\bar{z}^G, \bar{X}^G, \bar{Y}^G, \bar{Z}^G)$ that represents a near embedding. To obtain the actual near embedding, we need to compute an embedding σ of the graph G_0 in a surface Σ of Euler genus at most γ with the desired properties. We modify the graph G_0 by adding walls W_i of sufficient height and attaching them to the cycle with edge set Z_i^G in a similar way as we did in the proof of Lemma 5.3 and then use Mohar's linear time algorithm for computing embeddings in a surface [20]. □

6 Constructing a tree-decomposition over α-near embeddable graphs

Lemma 6.1. *Let $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ be nonnegative integers with $\beta > \alpha_1 + max\{3, 2\alpha_3 + 1\}$. Then there is a quadratic time algorithm that, given an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposable graph G, computes an $(\alpha_1 + 3\beta - 2, \alpha_2, 2\alpha_3 + 2, \beta, \gamma)$-decomposition of G.*

Proof. The proof is a standard construction that goes back to [27]. An algorithmic version of the construction has recently been applied in [15].

We describe a recursive algorithm that, given a locally $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposable graph G and a set $U \subseteq V(G)$ of size $\leq 3\beta - 2$, constructs a $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition (T, Y) of the graph $\overline{G}$ obtained from G by adding edges between any two vertices in U. We view T as a rooted tree and assume that $U \subseteq Y_r$ for the root r. (As U is a clique in $\overline{G}$, there is some node r such that $U \subseteq Y_r$, and we may assume r to be the root.)

So let G be a locally $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposable graph and $U \subseteq V(G)$ with $|U| \leq 3\beta - 2$. If $|G| \leq 4\beta - 3$, we let T be a one node tree only consisting of the root r, and we let $Y_r = V(G)$. In the following, we assume that $|G| > 4\beta - 3$. We let $U' \supseteq U$ be a subset of $V(G)$ of size $|U'| = 3\beta - 2$. To simplify the notation, we assume that $|U| = 3\beta - 2$ and hence $U' = U$.

If U is not unbreakable in G, we let (A, B) be a separation of G of order $< \beta$ that breaks U. Such a separation can be computed in time $2^{O(\beta)}n$ (see [15]). Let $S := V(A \cap B)$. As (A, B) breaks U, we have $|(V(A) \cap U) \cup S| < 3\beta - 2$. If $V(A) \setminus (U \cup S) \neq \emptyset$, we let u_A be an arbitrary vertex in this set, and we let $U_A = (V(A) \cap U) \cup S \cup \{u_A\}$; otherwise we let $U_A = (V(A) \cap U) \cup S$. We define U_B similarly. We recursively decompose (A, U_A) and (B, U_B), and let (T_A, Y_A) and (T_B, Y_B) be the resulting decompositions. To construct the tree T, we take the disjoint union of the trees T_A and T_B. We add a new root r and edges from r to the roots of the trees T_A and T_B. We let $Y_r = U \cup S$. For each node $t \in V(T_A)$ we let $Y_t = (Y_A)_t$ and for each node $t \in V(T_B)$ we let $Y_t = (Y_B)_t$.

If U is unbreakable, we use the algorithm of Theorem 5.2 to compute a $\mathfrak{T}_U$-central nice $(\alpha_1, \alpha_2', \alpha_3)$-near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ of G in a surface Σ of Euler genus at most γ, for some $\alpha_2' \leq \alpha_2$. Such an embedding exists because G is locally $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposable. We assume that $\mathcal{V} = \{(G_i, \Omega_i) \mid 1 \leq i \leq \alpha_2'\}$ and $\mathcal{W} = \{(G_i, \Omega_i) \mid \alpha_2' + 1 \leq i \leq m\}$.

For $1 \leq i \leq \alpha_2'$, we use the algorithm of Lemma 3.2 to compute a linear decomposition $(X_{ij})_{0 \leq j < \ell_i}$ of (G_i, Ω_i) of depth at most $2\alpha_3+2$. Let $x_{i0}, \ldots, x_{i(\ell-1)}$ be an enumeration of $V(\Omega_i)$ such that $x_{ij} \in X_{ij}$ and $x_{ij} = \Omega_i^j(x_{i0})$. Let $X_{i0}' = (X_{i0} \cap X_{i1}) \cup \{x_{i0}\}$, and for $1 \leq j < \ell_i$, let $X_{ij}' = (X_{i(j-1)} \cap X_{ij}) \cup (X_{ij} \cap X_{i(j+1)}) \cup \{x_{ij}\}$. Let $X_i = \bigcup_{j=0}^{\ell_i - 1} X_{ij}'$. Then for every connected component A of $G_i \setminus X_i$ there is a j such that $N^{G_i}(A) \subseteq X_{ij}'$. We let H_i be the graph obtained from $G_i[X_i]$ by adding an edge between any two nonadjacent vertices $v, w \in X_{ij}'$, for any j. Then $(X_{ij}')_{0 \leq j < \ell_i}$ is a linear decomposition of the society (H_i, Ω_i) of depth at most $2\alpha_3 + 2$ and width at most $4\alpha_3 + 5$.

Let $X_0 = V(G_0)$, and let $X = \bigcup_{i=0}^{\alpha_2'} X_i$. Then for every connected component C of $G \setminus (X \cup Z)$ we have $C \subseteq G_i$ for some $i \in [m]$. If $i \leq \alpha_2'$, there is a $j < \ell_i$ such that $N^G(C) \subseteq X_{ij}' \cup Z$ and thus $|N^G(C)| \leq \alpha_1 + 2\alpha_3 + 1$. If $i > \alpha_2'$ then $|N^G(C) \setminus Z| \leq 3$ and thus $|N^G(C)| \leq \alpha_1 + 3$. In both cases, $|N^G(C)| < \beta$. Let A be the graph with vertex set $V(A) = V(C) \cup (N^G(C) \cup Z)$ and edge set $E(A) = E(C) \cup \{vw \in E(G) \mid v \in V(C), w \in N^G(C) \cup Z\}$, and let B be the graph with vertex set $V(G) \setminus V(C)$ and edge set $E(G) \setminus E(A)$. Then (A, B) is a separation of G of order $< \beta$ and with $Z \subseteq V(A \cap B)$. Moreover, we have $A \setminus Z \subseteq G_i$ for some $i \in [n]$. As the near embedding $(\sigma, G_0, Z, \mathcal{V}, \mathcal{W})$ is $\mathfrak{T}_U$-central, this implies that $(B, A) \notin \mathfrak{T}_U$ and thus $|(V(A) \cap U) \cup V(A \cup B)| < 3\beta - 2$. Let $U' = (V(A) \cap U) \cup V(A \cap B)$. We recursively decompose (A, U') and obtain a tree decomposition (T_C, Y_C) of the graph $\overline{A}$ obtained from A by turning U' into a clique, and we may assume that $U' \supseteq U \cap V(A)$ is contained in the bag of the root of the tree T_C.

To construct the tree T, we take the disjoint union of the trees T_C for all connected components C of $G \setminus (X \cup Z)$. We add a new root r and edges from r to the roots of the trees T_C. We let $Y_r = X \cup Z \cup U$, and for each node $t \in V(T_C)$ we let $Y_t = (Y_C)_t$.

It remains to verify that the torso of the root has a near-embedding of the desired form. We let $H_0 = G_0'$ be the graph obtained from G_0 by adding an edge between any two nonadjacent vertices $v, w \in V(\Omega)$ for any $(J, \Omega) \in \mathcal{W}$, and we let H be the graph obtained from $H_0 \cup \bigcup_{i=1}^{\alpha_2'} H_i$ by adding all vertices in $Z \cup U$ and for each $z \in Z \cup U$ edges from z to all other vertices. Then H is a supergraph of the torso of our decomposition at r. Moreover, $(\sigma, H_0, Z \cup U, \{(H_i, \Omega_i) \mid 1 \leq i \leq \alpha_2'\}, \emptyset)$ is a nice $(\alpha_1 + 3\beta - 2, \alpha_2', 2\alpha_3 + 2)$-near embedding of H in Σ.

As it requires linear time to process each node of the tree, the overall running time of an algorithm computing the decomposition is quadratic in $|G|$. $\square$

Remark 6.1. *Note that in the proof of the lemma we needed the bound on the tree width of G only to guarantee that the decomposition can be computed in quadratic time. If we are not interested in an efficient algorithm, the construction shows how to obtain a decomposition for any locally decomposable graph and thus yields a proof of the Global Structure Theorem 3.2.*

We are now ready to prove our main theorem.

Theorem 6.1. (Main Theorem) *For all graphs R there are nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ and a quadratic time algorithm that, given a graph G that excludes R as a minor, computes an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of G.*

Proof. Let G be a graph that excludes R as a minor. Then G has an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition for a suitable choice of the parameters $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$.

By repeatedly applying Corollary 4.1 (of the Weak Structure Theorem) and the Extension Lemma 4.1, we can find a sequence $w_1, \ldots, w_m$ such that $\mathrm{tw}(G \setminus \{w_1, \ldots, w_m\}) \leq k$, for some $k = k(R)$, and that $G \setminus \{w_1, \ldots, w_{i-1}\}$ has an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition if any only if $G \setminus \{w_1, \ldots, w_i\}$ has. Furthermore, we can compute w_{i+1} from $w_1, \ldots, w_i$ in linear time, and we can compute a decomposition of $G \setminus \{w_1, \ldots, w_{i-1}\}$ from a decomposition of $G \setminus \{w_1, \ldots, w_i\}$ in linear time.

Our algorithm first computes $w_1, \ldots, w_m$, which takes quadratic time. Then it uses the algorithm of Lemma 6.1 to compute an $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$-decomposition of $G \setminus \{w_1, \ldots, w_m\}$, which also takes quadratic time. In the third step, for $i = m, m-1, \ldots, 1$ it extends the decomposition of $G \setminus \{w_1, \ldots, w_i\}$ to a decomposition of $G \setminus \{w_1, \ldots, w_{i-1}\}$. Eventually, this yields a decomposition of G. Each extension step takes linear time, so overall the extension requires quadratic time. $\square$

7 Conclusions

We give a quadratic time algorithm for computing graph minor decompositions. Given a graph G that excludes some fixed graph R as a minor, it computes a tree decomposition of G into pieces that are either "small" (of size bounded in terms of $|R|$) or "nearly embeddable" in a bounded genus surface (where the bound on the genus and the parameters associated with near embeddability depend on $|R|$). In addition, for all nearly embeddable pieces we can actually compute a near embedding, still within quadratic time. Our algorithm improves previously known algorithms for computing such decompositions and thus improves many results from algorithmic graph minor theory, which rely on the computation of such a decomposition. Moreover, and maybe even more importantly, our algorithm is significantly simpler than the previous ones.

Let us make two final remarks regarding easy extensions of our results that we omitted in this conference paper to keep the presentation simpler. First, our algorithm can be made uniform, that is, we can construct a single algorithm that given graphs G and R computes a decomposition of G provided G excludes R as a minor. (If G contains R as a minor, the algorithm may still succeed to compute a decomposition, or it may fail.) The running time of the algorithm is $f(|R|)|G|^2$ for some computable function f. And second, we can extend our algorithm such that it can be applied to arbitrary graphs, and if it fails to compute a decomposition (because the input graphs contains R as a minor), then it returns an R-minor in G.

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