On tree width, bramble size, and expansion

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Abstract

A *bramble* in a graph G is a family of connected subgraphs of G such that any two of these subgraphs have a nonempty intersection or are joined by an edge. The *order* of a bramble is the least number of vertices required to cover every subgraph in the bramble. Seymour and Thomas [8] proved that the maximum order of a bramble in a graph is precisely the tree width of the graph plus one. We prove that every graph of tree width at least k has a bramble of order $\Omega(k^{1/2}/\log^2 k)$ and size polynomial in n and k, and that for every k there is a graph G of tree width $\Omega(k)$ such that every bramble of G of order $k^{1/2+\varepsilon}$ has size exponential in n. To prove the lower bound, we establish a close connection between linear tree width and vertex expansion. For the upper bound, we use the connections between tree width, separators, and concurrent flows.

1 Introduction

Tree width is a fundamental graph invariant with many applications in graph structure theory and graph algorithms. Tree width has a dual characterisation in terms of brambles [6, 8]. A *bramble* in a graph G is a family of connected subgraphs of G such that any two of these subgraphs have a nonempty intersection or are joined by an edge. The *order* of a bramble is the least number of vertices required to cover all subgraphs in the bramble. Seymour and Thomas [8] proved that a graph has tree width k—that is, the minimum width of a tree decomposition of G is k—if and only if the maximum order of a bramble of G is $k+1$.

Such a dual characterisation of a graph invariant can be very useful in algorithmic or complexity theoretic applications. A bramble of order $k+1$ is a witness that the graph has tree width at least k. However, it is not a good characterization of tree width in the coNP sense for two reasons: (1) The number of subgraphs in the bramble is not necessarily polynomial in the size of the graph and (2) it is NP-hard to determine the order of a bramble. These problems are hardly surprising: It is NP-complete to decide whether the tree width of a graph is at most k, thus it seems highly unlikely that tree width has a coNP characterization. Therefore, we do not expect that these difficulties can be fully avoided.

Motivated by such considerations, in this note we address the question of how large brambles actually need to be. It will be important in the following to distinguish between the *size* of a bramble, that is, the number of subgraphs it consists of, and its order. It is a fairly straightforward consequence of the graph minor theorem [7] that there is a function f such that every graph of tree width at least k has a bramble of order $k+1$ and cardinality $f(k)$. We raise as an open question whether f can be bounded from above by an exponential function of k. Here we establish an exponential lower bound for this function f. Actually, we prove a stronger result that applies also for brambles with order somewhat smaller than $k+1$: There is a family $(G_k)_{k\geq 1}$ of graphs such that for every $\varepsilon > 0$ and every k, the tree width of G_k is at least k, and every bramble of G_k of order at least $\Omega(k^{1/2+\varepsilon})$ has size exponential in n_k, where n_k is the number of vertices of G_k. Conversely, we prove that every graph of tree width k has a bramble of order $\Omega(k^{1/2}/\log^2 k)$ and size polynomial in n and k.

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In order to avoid problem (2) described above, we introduce a simple lower bound on the order of the bramble and investigate how close it is to the order. The *depth* of a bramble is the maximum (taken over all vertices v) number of subgraphs in the bramble that contains vertex v; clearly, the order of a bramble cannot be less than the ratio of the size and depth. We show that this ratio is $O(k^{1/2})$ in every bramble for the graphs G_k mentioned in the previous paragraph. On the other hand, in our polynomial-sized bramble construction, not only the order is $\Omega(k^{1/2}/\log^2 k)$, but this holds even for the ratio of the size and the depth. In summary, every graph with tree width at least k has a polynomial-size bramble that certifies in an easily verifiable way that the tree width is $\Omega(k^{1/2}/\log^2 k)$, thus avoiding both problems (1) and (2) above. However, in general, brambles witnessing that the tree width is $\Omega(k^{1/2+\varepsilon})$ run into these problems.

To establish the lower bound on the bramble size, we need sparse graphs with tree width linear in the number of vertices. In Section 2, we observe that graphs with positive vertex expansion have this property, hence bounded-degree expander graphs can be used for the lower bound. Furthermore, we prove the following converse statement: If all graphs in a class $\mathscr{C}$ have tree width linear in the number of vertices, then they contain subgraphs of linear size (again in the number of vertices) with vertex expansion bounded from below by a constant. Therefore, large expansion is the only reason why the tree width of a graph can be linear in the number of vertices.

For the upper bound, we use the balanced separator characterization of tree width and an integrality gap result for separators. We use a probabilistic construction to turn a concurrent flow into a bramble. In [5], a similar approach is used to find an appropriate embedding in a graph with large tree width, and thereby proving an almost tight lower bound on the time complexity of binary constraint satisfaction (CSP) in terms of the tree width of the primal graph. In fact, our investigations of bramble size were partly motivated by possible applications such as [5]. The negative results of the current paper show that brambles cannot be used directly in these applications.

2 Tree width and vertex expansion

For every positive integer n, the set $\{1,\dots,n\}$ is denoted by $[n]$.

The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. For $X \subseteq V(G)$, the induced subgraph of G with vertex set X is denoted by $G[X]$, and we let $G \setminus X = G[V(G) \setminus X]$. For a set $F \subseteq E$, by $G - F$ we denote the graph $(V, E \setminus F)$.

A *tree decomposition* of a graph G is a pair (T,B), where T is a tree and B is a mapping that associates with every node $t \in V(T)$ a set $B_t \subseteq V(G)$ such that $G = \bigcup_{t \in V(T)} G[B_t]$, and for every $v \in V(G)$ the set $\{t \in V(T) \mid v \in B(t)\}$ is connected in T. The sets B_t, for $t \in V(T)$, are called the *bags* of the decomposition. The *width* of the decomposition is $\max\{|B_t| - 1 \mid t \in V(T)\}$, and the *tree width* of G, denoted by $\mathrm{tw}(G)$, is the minimum of the widths of all tree decompositions of G.

Let G be a graph. For a set $X \subseteq V(G)$, we let $S(X)$ (the *sphere* around X) be the set of all vertices in $V(G) \setminus X$ that are adjacent to a vertex in X. For every $\alpha \in [0,1]$, we define the *vertex expansion* of G with parameter α as the number

$$\mathrm{vx}_\alpha(G) = \min_{\substack{X \subseteq V(G) \\ 0 < |X| \leq \alpha \cdot |V(G)|}} \frac{|S(X)|}{|X|}$$

if $\alpha \cdot |V(G)| \geq 1$ and $\mathrm{vx}_\alpha(G) = 0$ otherwise.

Proposition 1. *Let $n \geq 1$ and $0 \leq \alpha \leq 1$. Then for every n-vertex graph G we have*

$$\mathrm{tw}(G) \geq \lfloor \mathrm{vx}_\alpha(G) \cdot (\alpha/2) \cdot n \rfloor . \tag{2.1}$$

Proof. Let (T,B) be a tree decomposition of width $k = \mathrm{tw}(G)$. Without loss of generality we may assume that T is a rooted tree such that for each node $t \in V(T)$,

- either t has two children u_1, u_2, and we have $B_t = B_{u_1} = B_{u_2}$,
- or t has one child u, and we have $|B_t \triangle B_u| = 1$ (here $\triangle$ denotes the symmetric difference),
- or t is a leaf.

Let r be the root of T. For every $t \in V(T)$, let T_t denote the subtree of T with root t. (More precisely, T_t is the induced subtree of T whose vertex set consists of all vertices u such that t occurs on the unique path from r to u.) Let $C_t = \bigcup_u B_u \setminus B_t$, where the union ranges over all $u \in V(T_t)$.

Without loss of generality we assume $\alpha < 1$, because if $\alpha = 1$ then $\mathrm{vx}_\alpha(G) = 0$, and (2.1) is trivially satisfied. We further assume that $\alpha \cdot n \geq 2$, because if $\alpha \cdot n < 2$ then $\mathrm{vx}_\alpha(G)$ is at most the minimum degree of G, which is known to be bounded by the tree width.

Case 1: $|C_r| \leq (\alpha/2) \cdot n$.

Observe first

$$\frac{\lfloor \alpha \cdot n \rfloor}{n} > \frac{\alpha \cdot n - 1}{n} = \alpha - \frac{1}{n} \geq \frac{\alpha}{2},$$

where the last inequality holds because $\alpha \cdot n \geq 2$. Hence

$$\mathrm{vx}_\alpha(G) \leq \frac{n - \lfloor \alpha \cdot n \rfloor}{\lfloor \alpha \cdot n \rfloor} = \frac{n}{\lfloor \alpha \cdot n \rfloor} - 1 < \frac{2}{\alpha} - 1.$$

Since $C_r = V(G) \setminus B_r$, this implies

$$\begin{aligned} k+1 \geq |B_r| = n - |C_r| &\geq \left(1 - \frac{\alpha}{2}\right) \cdot n & \text{(because } |C_r| \leq (\alpha/2) \cdot n\text{)} \\ &= \left(\frac{2}{\alpha} - 1\right) \cdot \frac{\alpha}{2} \cdot n \\ &> \mathrm{vx}_\alpha(G) \cdot \frac{\alpha}{2} \cdot n. \end{aligned}$$

Case 2: $(\alpha/2) \cdot n < |C_r| \leq \alpha \cdot n$.

Since $S(C_r) \subseteq B_r$, we have

$$k+1 \geq |B_r| \geq |S(C_r)| \geq \mathrm{vx}_\alpha(G) \cdot |C_r| > \mathrm{vx}_\alpha(G) \cdot \frac{\alpha}{2} \cdot n.$$

Case 3: $|C_r| > \alpha \cdot n$.

Then there exists a vertex $s \in V(T)$ such that $|C_s| > \alpha \cdot n$ and $|C_t| \leq \alpha \cdot n$ for all children t of s. Let s be such a vertex, and let t be the child of s for which $|C_t|$ is maximum. Then

$$\frac{\alpha \cdot n}{2} < |C_t| \leq \alpha \cdot n. \tag{2.2}$$

To see this, we distinguish between s having one or two children. Note that s cannot be a leaf because $C_s \neq \emptyset$. If s has two children t and t', we have $B_s = B_t = B_{t'}$ and hence $C_s = C_t \cup C_{t'}$, which implies (2.2) because $|C_t| \geq |C_{t'}|$ and $|C_s| > \alpha \cdot n$. If t is the only child of s, then we have $|B_t \setminus B_s| = 1$ and hence $|C_t| = |C_s| - 1 > \alpha \cdot n - 1 \geq \alpha \cdot n/2$ because $\alpha \cdot n \geq 2$. Arguing as in Case 2, we have $S(C_t) \subseteq B_t$ and hence

$$k+1 \geq |B_t| \geq |S(C_t)| \geq \mathrm{vx}_\alpha(G) \cdot |C_t| > \mathrm{vx}_\alpha(G) \cdot \frac{\alpha}{2} \cdot n.$$

Hence all three cases yield

$$k+1 > \mathrm{vx}_\alpha(G) \cdot \frac{\alpha}{2} \cdot n,$$

which implies (2.1). □

Proposition 2. *Let $n \geq 1$, $\beta > 0$, and $0 < \alpha \leq 1/2$. Let G be an n-vertex graph such that $\mathrm{tw}(G) \geq \beta \cdot n$. Then there exists a subgraph $H \subseteq G$ with*

(1) $\mathrm{tw}(H) \geq (\beta/2) \cdot n$ *and hence* $|V(H)| \geq (\beta/2) \cdot n - 1$,

(2) $\mathrm{vx}_\alpha(H) \geq \beta/2$.

Proof. Since vx_α is monotone decreasing with respect to the parameter α, its suffices to prove the proposition for $\alpha = 1/2$. We inductively construct a sequence of subgraphs $H_0 \supseteq H_1 \supseteq \ldots \supseteq H_m$ of G. Let $H_0 = G$. Now suppose that we have constructed $H_0, \ldots, H_i$. Let $n_i = |V(H_i)|$. If $\mathrm{vx}_{1/2}(H_i) \geq \beta/2$, we let $m = i$ and stop the construction. Otherwise, there is a set $X \subseteq V(H_i)$ such that $|X| \leq n_i/2$ and $|S(X)| < (\beta/2) \cdot |X|$. Choose such a set X and let $H' = H_i[X]$ and $H'' = H_i \setminus X$.

Observe that $\mathrm{tw}(H_i) \leq \max\{\mathrm{tw}(H'), \mathrm{tw}(H'')\} + |S(X)|$: Given two tree decompositions of H' and H'', they can be joined together to a tree decomposition of H_i if each bag is extended with the set $S(X)$.

If $\mathrm{tw}(H') \geq \mathrm{tw}(H_i) - |S(X)|$, we let $H_{i+1} = H'$. Otherwise, we have $\mathrm{tw}(H'') \geq \mathrm{tw}(H_i) - |S(X)|$, and we let $H_{i+1} = H''$.

Note that in both cases we have $\mathrm{tw}(H_i) - \mathrm{tw}(H_{i+1}) \leq |S(X)| < (\beta/2) \cdot |X|$. Moreover, letting $n_{i+1} = |V(H_{i+1})|$ we have

$$n_i - n_{i+1} \geq |X|.$$

This follows from $|X| \leq n_i/2$ if $H_{i+1} = H'$ and is trivial if $H_{i+1} = H''$. Thus if in the $(i+1)$-th step of the construction, the tree width of the graph is reduced by k then the number of vertices is reduced by at least $(2/\beta) \cdot k$.

Let $H = H_m$. By the construction, we have $\mathrm{vx}_{1/2}(H) \geq \beta/2$. We claim that $\mathrm{tw}(H) \geq \mathrm{tw}(G)/2 \geq (\beta/2) \cdot n$. This follows from the fact that whenever the tree width is reduced by k in a step of the construction, the number of vertices is reduced by $(2/\beta) \cdot k$. Hence to reduce the tree width by more than $\mathrm{tw}(G)/2$, we would have to reduce the number of vertices by more than

$$\frac{2}{\beta} \cdot \frac{\mathrm{tw}(G)}{2} \geq \frac{\beta \cdot n}{\beta} = n,$$

which is impossible. □

The two propositions immediately imply the following result:

Theorem 3. *For every class $\mathscr{C}$ of graphs and every α with $0 < \alpha \leq 1/2$, the following statements are equivalent:*

(1) There is a constant $\beta > 0$ such that $\mathrm{tw}(G) \geq \beta \cdot |V(G)|$ for every $G \in \mathscr{C}$.

(2) There are constants $\gamma_1, \gamma_2 > 0$ such that every graph $G \in \mathscr{C}$ has a subgraph H such that $|V(H)| \geq \gamma_1 \cdot |V(G)|$ and $\mathrm{vx}_\alpha(H) \geq \gamma_2$.

□

A variant of Proposition 2, which can be proved using the same ideas, is the following proposition. It was suggested by S. Thomassé:

Proposition 4. *Let G be a graph and $\beta = \frac{\mathrm{tw}(G)}{|V(G)|}$, and suppose that for all proper subgraphs $H \subset G$ it holds that*

$$\frac{\mathrm{tw}(H)}{|V(H)|} < \beta.$$

Then $\mathrm{vx}_{1/2}(G) \geq \beta$.

Proof. Let $n = |V(G)|$. Suppose for contradiction that $\mathrm{vx}_{1/2}(G) < \beta$, and let $X \subseteq V(G)$ such that $|X| \leq n/2$ and $|S(X)|/|X| < \beta$. Then $\mathrm{tw}(G) \leq \max\{\mathrm{tw}(G[X]), \mathrm{tw}(G \setminus X)\} + |S(X)|$ by the same argument as in the proof of Proposition 2.

Case 1: $\mathrm{tw}(G) \leq \mathrm{tw}(G[X]) + |S(X)|$.

Then

$$\begin{aligned}
\beta \cdot n = \mathrm{tw}(G) &\leq \mathrm{tw}(G[X]) + |S(X)| \\
&< \beta \cdot |X| + \beta \cdot |X| && \text{(because } \mathrm{tw}(G[X])/|X| < \beta \text{ and } |S(X)|/|X| < \beta) \\
&\leq \beta \cdot n && \text{(because } |X| \leq n/2),
\end{aligned}$$

which is a contradiction.

Case 2: $\mathrm{tw}(G) \leq \mathrm{tw}(G \setminus X) + |S(X)|$.

Then

$$\beta \cdot n = \mathrm{tw}(G) \leq \mathrm{tw}(G \setminus X) + |S(X)| < \beta \cdot (n - |X|) + \beta \cdot |X| = \beta \cdot n,$$

again a contradiction. □

It is well known that there are families of graphs of bounded degree and positive vertex expansion; examples are random regular graphs. We state the following without proof (see [4] for a proof):

Theorem 5. *Let $d \geq 3$. Then for every $\varepsilon > 0$ there is an $\alpha > 0$ and a family $(G_n)_{n\geq 1}$ of d-regular graphs such that*

$$\mathrm{vx}_\alpha(G_n) \geq d - 1 - \varepsilon \quad \textit{for all } n \geq 1.$$

The *line graph* $L(G)$ of G contains one edge for each vertex of G, and the vertices of $L(G)$ are adjacent if and only if the corresponding two edges share an endpoint in G. Let us denote by L_k the line graph of the complete graph on k vertices (thus L_k has $\binom{k}{2}$ vertices). We show that L_k has positive vertex expansion, hence its tree width is linear in the number of vertices, i.e., $\Theta(k^2)$. Line graphs of cliques form an essential role in the embedding technique of [5] and implicitely in the upper bound of Section 3.

Lemma 6. *For every $k \geq 3$, $\mathrm{vx}_{1/2}(L_k) \geq 2\sqrt{2} - 2 + O(1/k)$.*

Proof. Let $v_{\{1,2\}}$, $v_{\{1,3\}}$ $\ldots$, $v_{\{k-1,k\}}$ be the $\binom{k}{2}$ vertices of L_k, where $v_{\{i_1,i_2\}}$ and $v_{\{j_1,j_2\}}$ are connected if and only if $\{i_1,i_2\} \cap \{j_1,j_2\} \neq \emptyset$. Let $X \subseteq V(L_k)$ be a set minimizing $|S(X)|/|X|$. Let $Y \subseteq \{1,2,\ldots,k\}$ be $\bigcup_{v_{\{i,j\}} \in X} \{i,j\}$. Observe that if $i,j \in Y$, then $v_{\{i,j\}} \in X \cup S(X)$; if $i \in Y$, $j \in \{1,\ldots,k\} \setminus Y$, then $v_{\{i,j\}} \in S(X)$. We consider two cases.

Case 1: $|Y| < k/\sqrt{2} + 1$.

In this case

$$\frac{|S(X)|}{|X|} \geq \frac{|Y|(k-|Y|)}{\binom{|Y|}{2}} \geq \frac{|Y|(k-|Y|)}{|Y|^2/2} \geq \frac{2k}{|Y|} - 2 > \frac{2k}{k/\sqrt{2}+1} - 2 = 2\sqrt{2} - 2 + O(1/k).$$

Case 2: $|Y| \geq k/\sqrt{2} + 1$. Since $|X| \leq |V(L_k)|/2 = k(k-1)/4$ and

$$\binom{|Y|}{2} \geq \frac{(k/\sqrt{2}+1-1)(k/\sqrt{2}+1)}{2} \geq \frac{k(k-1)}{4} \geq |X|,$$

there are at least $\binom{|Y|}{2} - k(k-1)/4 \geq 0$ vertices $v_{\{i,j\}} \in S(X)$ with $i,j \in Y$. Together with the $|Y|(k-|Y|)$ vertices of $S(X)$ of the the form $v_{\{i,j\}}$ with $i \in Y$, $j \notin Y$, we have that

$$\frac{|S(X)|}{|X|} \geq \frac{\binom{|Y|}{2} - k(k-1)/4 + |Y|(k-|Y|)}{k(k-1)/4}.$$

This expression is a concave function of $|Y|$ for a fixed $k \geq 3$, hence the minimum is attained either for $|Y| = k$ or $|Y| = \lceil k/\sqrt{2}+1 \rceil$. If $|Y| = k$, then $S(X) = V(L_k) \setminus X$, hence $|S(X)|/|X| \geq 1$. Substituting $|Y| = \lceil k/\sqrt{2}+1 \rceil$ into the bound above gives

$$\frac{|S(X)|}{|X|} \geq \frac{k^2/4 - k^2/4 + k^2/\sqrt{2} - k^2/2 + O(k)}{k^2/4 + O(k)} = 4(1/\sqrt{2} - 1/2) + O(1/k) = 2\sqrt{2} - 2 + O(1/k).$$

□

Corollary 7. *The tree width of L_k is at least $k^2 \cdot (\sqrt{2}-1)/4 + O(k)$.* □

3 Bramble size

Let us state the main definitions concerning brambles more formally. Let G be a graph. We say that two subgraphs $A,B \subseteq G$ *touch* if either $V(A) \cap V(B) \neq \emptyset$ or there is an edge $e \in E(G)$ that is incident with a vertex of A and a vertex of B. A set $X \subseteq V(G)$ *covers* a subgraph $B \subseteq G$ if $X \cap V(B) \neq \emptyset$, and X covers a family $\mathscr{B}$ of subgraphs of G if it covers all graphs $B \in \mathscr{B}$. A *bramble* of G is a family $\mathscr{B}$ of connected subgraphs of G any two of which touch. For example, for every connected graph G, the set of all connected subgraphs with more than $|V(G)|/2$ vertices is a bramble of G. The *size* of $\mathscr{B}$ is simply $|\mathscr{B}|$. The *order* of $\mathscr{B}$ is the least k such that there is a set X with $|X| = k$ that covers $\mathscr{B}$. The *depth* of $\mathscr{B}$ is $\max_{v \in V(G)} |\{B \in \mathscr{B} \mid v \in B\}|$. It is easy to see that the order of the bramble is at least the ratio of the size and the depth, since the depth is the maximum number of sets that a vertex can cover. The *bramble number* of a graph G is the maximum of the orders of all brambles of G. Seymour and Thomas [8] proved that the bramble number of a graph is its tree width plus one.

The main result of the section is the following theorem, which shows that if we want to find a bramble whose size is polynomial in the number of vertices, then the maximum order we can expect is roughly the square root of the tree width:

Theorem 8.

(1) Every n-vertex graph G of tree width k has a bramble of order $\Omega(k^{1/2}/\log^2 k)$ and size $O(k^{3/2} \cdot \ln n)$.

(2) There is a family $(G_k)_{k \geq 1}$ of graphs such that:

- $|V(G_k)| = O(k)$ *and* $|E(G_k)| = O(k)$ *for every* $k \geq 1$*;*
- $\mathrm{tw}(G_k) \geq k$ *for every* $k \geq 1$*;*
- *for every* $\varepsilon > 0$ *and* $k \geq 1$*, every bramble of* G_k *of order at least* $k^{1/2+\varepsilon}$ *has size at least* $2^{\Omega(k^{\varepsilon})}$*;*
- *in every bramble of* G_k*, the ratio of the size and the depth is* $O(k^{1/2})$.

The proof of the first part of Theorem 8 is based on the characterization of tree width by balanced separators and uses a result of Feige et al. [2] on the linear programming formulation of separation problems. A similar approach is used in [5] to find an embedding in a graph with large tree width; some of the arguments are repeated here for the convenience of the reader. A *separator* of a graph G is a partition of the vertices into three classes (A,B,S) $(S \neq \emptyset)$ such that there is no edge between A and B. A k-separator is a separator (A,B,S) with $|S| = k$. Given a set W of vertices and a separator (A,B,S), we say that S is a *balanced separator* (with respect to W) if $|W \cap C| \leq |W|/2$ for every connected component C of $G \setminus S$. The tree width of a graph is closely connected with the existence of balance separators:

Lemma 9 ([6], [3, Section 11.2])**.**

(1) If $G(V,E)$ has tree width greater than $3k$, then there is a set $W \subseteq V$ of size exactly $2k+1$ having no balanced k-separator.

(2) If $G(V,E)$ has tree width at most k, then every $W \subseteq V$ has a balanced $(k+1)$-separator.

The *sparsity* of the separator (A,B,S) (with respect to W) is defined as

$$\alpha^W(A,B,S) = \frac{|S|}{|(A \cup S) \cap W| \cdot |(B \cup S) \cap W|}.$$

We denote by $\alpha^W(G)$ the minimum of $\alpha^W(A,B,S)$ for every separator (A,B,S). It is easy to see that for every connected G and nonempty W, $1/|W|^2 \leq \alpha^W(G) \leq 1/|W|$. For our applications, we need a set W such that the sparsity is close to the maximum possible, i.e., $\Omega(1/|W|)$. The following lemma shows that the non-existance of a balanced separator can guarantee the existence of such a set W:

Lemma 10. *If $|W| = 2k+1$ and W has no balanced k-separator in a graph G, then $\alpha^W(G) \geq 1/(4k+1)$.*

Proof. Let (A,B,S) be a separator of sparsity $\alpha^W(G)$; without loss generality, we can assume that $|A\cap W| \geq |B\cap W|$, hence $|B\cap W| \leq k$. If $|S| > k$, then $\alpha^W(A,B,S) \geq (k+1)/(2k+1)^2 \geq 1/(4k+1)$. If $|S| \geq |(B\cup S)\cap W|$, then $\alpha^W(A,B,S) \geq 1/|(A\cup S)\cap W| \geq 1/(2k+1)$. Assume therefore that $|(B\cup S)\cap W| \geq |S|+1$. Let S' be a set of $k-|S| \geq 0$ arbitrary vertices of $W\setminus(S\cup B)$. We claim that $S\cup S'$ is a balanced separator of W. Suppose that there is a component C of $G\setminus(S\cup S')$ that contains more than k vertices of W. Component C is either a subset of A or B. However, it cannot be a subset of B, since $|B\cap W| \leq k$. On the other hand, $|(A\setminus S')\cap W|$ is at most $2k+1-|(B\cup S)\cap W| - |S'| \leq 2k+1-(|S|+1)-(k-|S|) \leq k$. □

Remark 11. Lemma 10 does not remain true in this form for larger W. For example, let K be a clique of size $3k+1$, let us attach k degree one vertices to a distinguished vertex x of K, and let us attach a degree one vertex to every other vertex of K. Let W be the set of these $4k$ degree one vertices. It is not difficult to see that W has no balanced k-separator. On the other hand, $S=\{x\}$ is a separator with sparsity $1/(k(3k+1))$, hence $\alpha^W(G) = O(1/k^2)$.

Let $W=\{w_1,\ldots,w_r\}$ be a set of vertices. A *concurrent vertex flow of value* ε is a collection of $|W|^2$ flows such that for every ordered pair $(u,v)\in W\times W$, there is a flow of value ε between u and v, and the total amount of flow going through each vertex is at most 1. A flow between u and v is a weighted collection of $u-v$ paths. A $u-v$ path contributes to the load of vertex u, of vertex v, and of every vertex between u and v on the path. In the degenerate case when $u=v$, vertex $u=v$ is the only vertex where the flow between u and v goes through, that is, the flow contributes to the load of only this vertex.

The maximum concurrent vertex flow can be expressed as a linear program the following way. For $u,v\in W$, let $\mathscr{P}_{uv}$ be the set of all $u-v$ paths in G, and for each $p\in\mathscr{P}_{uv}$, let variable p^{uv} denote the amount of flow that is sent from u to v along p. Consider the following linear program:

$$
\begin{aligned}
&\text{maximize } \varepsilon \\
&\text{s. t.} \\
&\sum_{p\in\mathscr{P}_{uv}} p^{uv} \geq \varepsilon && \forall u,v\in W \\
&\sum_{(u,v)\in W\times W}\ \sum_{p\in\mathscr{P}_{uv}: w\in p} p^{uv} \leq 1 && \forall w\in V \qquad \text{(LP1)} \\
&p^{uv} \geq 0 && \forall u,v\in V, p\in\mathscr{P}_{uv}
\end{aligned}
$$

The dual of this linear program can be written with variables $\{\ell_{uv}\}_{u,v\in W}$ and $\{s_v\}_{v\in V}$ the following way:

$$
\begin{aligned}
&\text{minimize } \sum_{v\in V} s_v \\
&\text{s. t.} \\
&\sum_{w\in p} s_w \geq \ell_{uv} && \forall u,v\in W, p\in\mathscr{P}_{uv}\ (*) \\
&\sum_{(u,v)\in W\times W} \ell_{uv} \geq 1 && (**) \qquad \text{(LP2)} \\
&\ell_{uv} \geq 0 && \forall u,v\in W \\
&s_w \geq 0 && \forall w\in V
\end{aligned}
$$

We show that if there is a separator (A,B,S) with sparsity $\alpha^W(A,B,S)$, then (LP2) has a solution with value at most $\alpha^W(A,B,S)$. Set $s_v = \alpha^W(A,B,S)/|S|$ if $v\in S$ and $s_v=0$ otherwise; the value of such a solution is clearly $\alpha^W(A,B,S)$. For every $u,v\in W$, set $\ell_{uv} = \min_{p\in\mathscr{P}_{uv}}\sum_{w\in p} s_v$ to ensure that inequalities (*) hold. To see that (**) holds, notice first that $\ell_{uv} \geq \alpha^W(A,B,S)/|S|$ if $u\in A\cup S$, $v\in B\cup S$, as every $u-v$ path has to go through at least one vertex of S. Furthermore, if $u,v\in S$ and $u\neq v$, then $\ell_{uv} \geq 2\alpha^W(A,B,S)/|S|$ since in this case a $u-v$ paths meets S in at least two vertices. The expression $|(A\cup S)\cap W|\cdot|(B\cup S)\cap W|$ counts the number of ordered pairs (u,v) satisfying $u\in(A\cup S)\cap W$ and $v\in(B\cup S)\cap W$, such that pairs with $u,v \in S \cap W$, $u \neq v$ are counted twice. Therefore,

$$\sum_{(u,v)\in W\times W} \ell_{uv} \geq (|(A\cup S)\cap W| \cdot |(B\cup S)\cap W|) \cdot \frac{\alpha^W(A,B,S)}{|S|} = 1,$$

which means that inequality (**) is satisfied.

The other direction is not true: a solution of (LP2) with value α does not imply that there is a separator with sparsity at most α. However, Feige et al. [2] proved that it is possible to find a separator whose sparsity is greater than that by at most a $O(\log|W|)$ factor:

Theorem 12 (Feige et al. [2]). *If (LP2) has a solution with value α, then there is a separator with sparsity $O(\alpha \log |W|)$.*

Now we are ready to prove the first part of Theorem 8. In the proof we use the following form of the Chernoff Bound to bound the probability of certain events:

Theorem 13 ([1]). *Let $X_1, X_2, \ldots, X_n$ be independent 0-1 random variables with $\Pr[X_i = 1] = p_i$. Denote $X = \sum_{i=1}^n X_i$ and $\mu = \mathrm{E}[X]$. Then*

$$\Pr[X \geq (1+\beta)\mu] \leq \begin{cases} \exp(-\beta^2\mu/3) \text{ for } 0 < \beta \leq 1, \\ \exp(-\beta^2\mu/(2+\beta)) \text{ for } \beta > 1. \end{cases}$$

Lemma 14. *Let $k \geq 2$, and let G be a graph of tree width greater than $3k$. Then G has a bramble of order $\Omega(\sqrt{k}/\log^2 k)$ and size $O(k^{3/2}\log n)$.*

Proof. Since G has tree width greater than $3k$, by Lemma 9, there is a subset W_0 of size at most $2k+1$ that has no balanced k-separator. By Lemma 10, $\alpha^{W_0}(G) \geq 1/(4k+1) \geq 1/(5k)$. Therefore, Theorem 12 implies that the dual linear program has no solution with value less than $1/(c_0 5k\log(2k+1))$, where c_0 is the constant hidden by the big O notation in Theorem 12. Let c be a constant such that $1/(c_0 5k \log(2k+1)) \geq 1/(ck\ln k)$ for $k \geq 2$ (here $\ln k$ denotes the natural logarithm of k). By linear programming duality, there is a concurrent flow of value at least $\alpha := 1/(ck\ln k)$ connecting the vertices of W_0; let p^{uv} be a corresponding solution of (LP1).

Let $W \subseteq W_0$ be a subset of k vertices. For each pair of vertices $(u,v) \in W \times W$, we define a probability distribution on $\mathscr{P}_{uv}$ by setting the probability of $p \in \mathscr{P}_{uv}$ to be

$$\frac{p^{uv}}{\sum_{p'\in\mathscr{P}_{uv}}(p')^{uv}} \leq \frac{p^{uv}}{\alpha}.$$

We construct a bramble $\mathscr{B}$ containing $\lfloor k^{3/2}\rfloor\lfloor \ln n\rfloor$ sets. Set $d = \lfloor k^{3/2}\rfloor$ and $s := \lfloor\sqrt{k}\ln k\rfloor$. Let us select uniformly and independently d random subsets $S_1,\ldots,S_d \subseteq W$, each of size s. For each S_i, let us select uniformly at random a vertex $z_i \in W \setminus S_i$. For each S_i, we construct a collection $\mathscr{B}_i$ of $\lfloor \ln n\rfloor$ sets $B_{i,1},\ldots,B_{i,\lfloor \ln n\rfloor}$ the following way. If $S_i = \{u_{i,1},\ldots,u_{i,s}\} \subseteq W$, then $B_{i,j}$ is constructed by selecting a random path from each of $\mathscr{P}_{z_iu_{i,1}}$, $\mathscr{P}_{z_iu_{i,2}}$, $\ldots$, $\mathscr{P}_{z_iu_{i,s}}$ according to the probability distribution defined above and taking the union of these s paths. Clearly, $B_{i,j}$ is a connected set: each path contains z_i.

We claim that with high probability, the sets in $\mathscr{B} = \cup_{i=1}^d \mathscr{B}_i$ form a bramble. If S_i and $S_{i'}$ have nonempty intersection, then the sets $B_{i,j}$ and $B_{i',j'}$ have nonempty intersection as well. The probability that random subsets S_i and $S_{i'}$ are disjoint is at most

$$\frac{\binom{k-s}{s}}{\binom{k}{s}} = \frac{k-s}{k}\cdot\frac{k-s-1}{k-1}\cdot\cdots\cdot\frac{k-2s+1}{k-s+1} \leq \left(1-\frac{s}{k}\right)^s \leq \exp\left(-\frac{s^2}{k}\right) \leq \exp(-4\ln k) \leq \frac{1}{k^4},$$

if k is sufficiently large. There are $\binom{d}{2} \leq \lfloor k^{3/2}\rfloor^2$ pairs $\{S_i, S_{i'}\}$, thus by the union bound, the S_i's pairwise touch by probability at least $1-1/k$.

To bound the order of the bramble $\mathscr{B}$, we show that with high probability, each vertex is contained in at most $24ck\ln^2 k\cdot\ln n$ sets of $\mathscr{B}$ (where c is the universal constant defined at the beginning of the proof). First, we show that the following event holds with high probability:

(E1) For every $x,y \in W$, there are at most $12\ln k$ values of i such that $z_i = x$ and $y \in S_i$.

Fixing x, y, and i, let us bound the probability that $z_i = x$ and $y \in S_i$. If $x = y$, then this event has probability 0; otherwise, its probability is exactly $(1/k)\cdot(s/(k-1))$. Thus

$$\Pr\left(z_i = x, y \in S_i\right) \le \frac{1}{k}\cdot\frac{s}{k-1} \le \frac{1}{k}\cdot\frac{2\sqrt{k}\ln k}{k} = \frac{2\ln k}{k^{3/2}}.$$

Fixing x and y, the probability that this happens for more than $12\ln k$ values of i (i.e., more than 6 times the expected number of times) can be bounded using the Chernoff Bound (Theorem 13 with $\beta = 5$):

$$\Pr\left(|i : z_i = x, y \in S_i| \ge 12\ln k\right) \le e^{-\frac{50}{7}\ln k} \le \frac{1}{k^3}.$$

There are k^2 pairs $x,y \in W$ thus by the union bound, event (E1) holds with probability at least $1-1/k$.

For a vertex v, and $x,y \in W$, let $\gamma_{x,y}(v)$ be the total weight of the $x-y$ paths going through v in the solution for (LP1), that is, $\gamma_{x,y} := \sum_{p\in\mathscr{P}_{xy}: v\in p} p^{xy}$. Let us fix the sets $S_1, \ldots, S_d$ and the vertices $z_1, \ldots, z_d$, and assume that (E1) holds. Let $S_i = \{u_{i,1},\ldots,u_{i,s}\}$. As $B_{i,j}$ is the union of random paths from $\mathscr{P}_{z_i,u_{i,1}}, \ldots, \mathscr{P}_{z_i,u_{i,s}}$, the probability that $B_{i,j}$ contains v is at most $\sum_{\ell=1}^{s}\gamma_{z_i,s_{i,\ell}}(v)/\alpha$. Thus the expected number of sets in $\mathscr{B}_i$ that contain v is at most $\lfloor \ln n\rfloor \sum_{\ell=1}^{s}\gamma_{z_i,s_{i,\ell}}(v)/\alpha$. Summing for every $1 \le i \le d$ and using the assumption that (E1) holds, the expected number of sets that contain a given v is at most

$$\lfloor \ln n\rfloor \sum_{i=1}^{d}\sum_{\ell=1}^{s}\gamma_{z_i,s_{i,\ell}}(v)/\alpha \le 12\ln k\cdot\lfloor \ln n\rfloor \sum_{x,y\in W}\gamma_{x,y}(v)/\alpha \le 12\ln k\cdot\lfloor \ln n\rfloor/\alpha \le 12ck\ln^2 k\cdot \ln n.$$

If $S_1, \ldots, S_d$ are fixed, the number of sets that contain a vertex v can be expressed as the sum of $d\lfloor \ln n\rfloor$ independent 0-1 random variables. Hence we can apply the Chernoff Bound (Theorem 13 with $\beta = 1$) to show that the probability that vertex v is covered by too many sets is at most

$$\Pr\left(|B\in\mathscr{B} : v\in B| \ge 24ck\ln^2 k\cdot\ln n\right) \le \exp\left(-4k\ln^2 k\cdot \ln n\right) \le \frac{1}{n^2},$$

if k is sufficiently large. Thus by the union bound, with high probability every vertex v is contained in at most $24ck\ln^2 k\cdot\ln n$ sets of $\mathscr{B}$. Therefore, bramble $\mathscr{B}$ can be covered only with at least

$$\lfloor k^{3/2}\rfloor\lfloor \ln n\rfloor/(24ck\ln^2 k\cdot\ln n) = \Omega(\sqrt{k}/\log^2 k)$$

vertices, which gives the required lower bound on the order. □

Remark 15. The size of the bramble in Lemma 14 depends not only on k, but on n as well. Therefore, this construction does not answer the stronger form of the question when we require a bound on the size that depends only on k. Using completely different techniques, we were able to prove a version of Lemma 14 where the order is only $\Omega(k^{1/3})$, but the size is $O(k^{2/3})$ and hence independent of n.

The second part of Theorem 8 is based on the observation that in bounded-degree graphs, every bramble with order significantly greater than $\sqrt{n}$ must have exponential size. There are bounded-degree graphs with tree width linear in n (e.g., graphs with positive vertex expansion); for such graphs the order of a polynomial-size bramble is at most $\sqrt{n}$.

Lemma 16. *Let G be an n-vertex graph of maximum degree d, and let $\mathscr{B}$ be a bramble in G of order greater than $\lceil c\cdot n^{1/2+\varepsilon}\rceil$ for some $c,\varepsilon > 0$. Then*

$$|\mathscr{B}| \ge \exp\left(c\cdot n^{\varepsilon}/(d+1)\right).$$

Proof. Suppose $\mathscr{B}$ has a set B of cardinality at most $c\cdot n^{1/2}/(d+1)$. Let S contain every vertex of B and every vertex adjacent to a vertex in B. Set S covers $\mathscr{B}$, since B touches every set in $\mathscr{B}$. However, the cardinality of S is at most $c\cdot n^{1/2}$, contradicting the assumption that the order of $\mathscr{B}$ is greater than $\lceil c\cdot n^{1/2+\varepsilon}\rceil$. Thus we can assume that every $B\in\mathscr{B}$ has cardinality at least $c\cdot n^{1/2}/(d+1)$.

Let $\ell := \lceil n^{1/2+\varepsilon} \rceil$. We choose vertices $v_1, \ldots, v_\ell$ independently uniformly at random. For $B \in \mathscr{B}$ and $i \in [\ell]$, we let X_i^B be the random variable that is 1 if $v_i \in B$ and 0 otherwise. Then

$$\Pr(X_i^B = 1) = \frac{|V(B)|}{|V(G)|} \geq \frac{c \cdot n^{1/2}/(d+1)}{n} = \frac{c \cdot n^{-1/2}}{(d+1)}.$$

Hence

$$\Pr\left(\sum_{i=1}^{\ell} X_i^B = 0\right) = \left(1 - \frac{c \cdot n^{-1/2}}{(d+1)}\right)^{\ell} \leq \exp\left(-\frac{c \cdot n^{-1/2}}{(d+1)} \cdot \ell\right) \leq \exp\left(-\frac{c \cdot n^{\varepsilon}}{(d+1)}\right)$$

and thus

$$\begin{aligned}\Pr\left(\{v_1, \ldots, v_\ell\} \text{ does not cover } \mathscr{B}\right) &= \Pr\left(\exists B \in \mathscr{B} : \sum_{i=1}^{\ell} X_i^B = 0\right) \\ &\leq \sum_{B \in \mathscr{B}} \Pr\left(\sum_{i=1}^{\ell} X_i^B = 0\right) \\ &\leq m \cdot \exp\left(-\frac{c \cdot n^{\varepsilon}}{(d+1)}\right)\end{aligned}$$

Since the order of $\mathscr{B}$ is greater than ℓ, we know that the last probability must be 1. Hence

$$1 \leq m \cdot \exp\left(-c \cdot n^{\varepsilon}/(d+1)\right),$$

which implies $m \geq \exp\left(c \cdot n^{\varepsilon}/(d+1)\right)$. □

Lemma 17. *Let G be an n-vertex graph of maximum degree d, and let $\mathscr{B}$ be a bramble in G. Then the ratio of the depth and the size of $\mathscr{B}$ is at most $(d+1)n^{1/2}$.*

Proof. Suppose first that $\mathscr{B}$ has a set B of cardinality at most $n^{1/2}/(d+1)$. As in the proof Lemma 16, this implies that the order of $\mathscr{B}$ is at most $n^{1/2}$, which further implies that ratio of the size and depth is also at most $n^{1/2}$. Thus we can assume that every $B \in \mathscr{B}$ has cardinality at least $n^{1/2}/(d+1)$. It follows that the depth of $\mathscr{B}$ is at least $(|\mathscr{B}|n^{1/2}/(d+1))/n = |\mathscr{B}|/((d+1)n^{1/2})$, hence the ratio of the size and the depth is at most $(d+1)n^{1/2}$. □

Proof of Theorem 8. Part (1) follows from Lemma 14. Part (2) follows from Proposition 1, Theorem 5, Lemma 16, and Lemma 17. □

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